Theory of fundamental interactions

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The course is divided into 3 parts:

1) Recap on QFT
2) Renormalization theory (in QED)
3) Spontaneus symmetry breaking

Exam: oral with 2/3 questions

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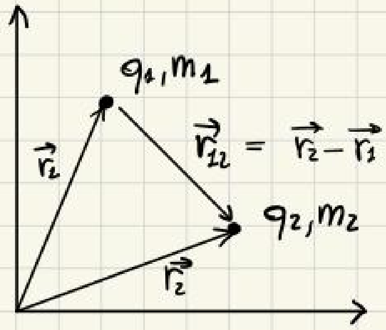
Part 1

4 FUNDAMENTAL INTERACTIONS: $\underbrace{\text{Gravity, EM}}_{\text{Long range}}$, $\underbrace{\text{Weak, Strong}}_{\text{Short range}}$

CONSIDERATIONS: (between EM and Gravity)



$$\begin{cases}\vec{F}_{12\,EM} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{r^2}\,\hat{r}_{12} \\ \vec{F}_{12\,G} = -G\,\dfrac{m_1 m_2}{r^2}\,\hat{r}_{12}\end{cases} \rightarrow \text{Very similar}$$

Let's take $2e^-$ $\begin{cases} m_1 = m_2 = m_e \simeq 0{,}5 \text{ MeV} \\ q_1 = q_2 = -e \end{cases}$ $c = \hbar = 1$

$$\longrightarrow \begin{cases}\vec{F}_{12\,EM} = \overset{\sim \frac{1}{137}}{\dfrac{e^2}{4\pi}}\,\dfrac{1}{r^2} \\ \\ \vec{F}_{12\,G} = \overset{\sim (4\cdot 10^{-23})^2}{\left(\dfrac{m_e}{M_{Plank}}\right)^2}\cdot\dfrac{1}{r^2}\end{cases}$$

$\rightarrow$ very small respect to α

$M_{Plank} = 1{,}2 \cdot 10^{19}$ GeV

This is a signal that G. and EM originates from the same principles.

What is QFT : theory which unifies Q.M. and S.R.

- What defines it? Lagrangian density.
- Why?
  1) The Lagrangian formalism works very well at class. level. Moreover is useful to go from classical physics to Q.M.
  2) Then if I have a $\mathcal{L}$ I know how to compute the main objects.
  3) It is very simple to implement symmetries

Why symmetries are so important?

There is an empirical evidence that nature has:

1) SIMPLICITY
2) UNIVERSALITY
3) BEAUTY

} The best language to implement these 3 principles is the language of symmetries or more precicely GROUP THEORY.

Simplicity ⟷ related to Noether's theorem : the presence of conserved quantities simplifies the description

Universality $\longleftrightarrow$ related to representation theory : able to apply the same symmetry principles to objects that are totally different between each other
E.g. : rotation in space

Beauty $\longleftrightarrow$ Lie groups are the most beauty objects interseption between algebra and diff. geometry)

The Lagrangian description howeever introduces redundancy : in order to make the formalism consistent we introduce more variable than the actual d.o.f. Examples : photon's d.o.f. , Dirac spinors d.o.f. This is the origin of the fact that computations in QFT are long.

HOW TO UNIFY Q.M WITH S.R.

Let's consider a free particle :

We Know how to find the Shrödinger equation:

$$\begin{cases} E \to i\partial_t \\ p \to -i\nabla \end{cases}$$ Quantization rule

- $E = \frac{p^2}{2m} \quad \rightarrow \quad i\partial_t\psi(\vec{x},t) = \frac{-1}{2m}\nabla^2\psi(\vec{x},t)$

We can do the same for the relativistic dispersion relation:

- $E^2 = p^2 + m^2 \quad \longrightarrow \quad (\Box + m^2)\phi(\vec{x},t) = 0$ Klein Gordon eq.

from which we can extrapolate the Klein Gordon Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2$$

and then promoting ϕ as $\hat{\phi}$ (by promoting the fourier coeff. to annihilation and creation op.) and imposing the commutation rule we can quantize the theory. The same we can do with Dirac. theory. This is not the approach that we want to follow because it hydes the generality of the construction of a Q.F.T., in particular the special role of symmetries.

OPERATORS

We Know that since $H = \frac{|P|^2}{2m} \rightarrow [H,P] = 0 \, ; \, [P^i, P^j] = 0$ (conservation of momentum)

So we have a complete description of the system if we consider the eigenstates of P. Therefore

$$P|p\rangle = p|p\rangle \quad ; \quad \langle q|p\rangle = \delta(p-q) \quad \int d^3p\, |p\rangle\langle p| = 1 \quad H|p\rangle = \frac{p^2}{2m}|p\rangle$$

$$\xrightarrow{\text{Any state}} |\psi\rangle = \underbrace{\int d^3p\, |p\rangle\langle p|\psi\rangle}_{\text{Superposition of momentum eigenstates}} = \int d^3p\, \hat{\psi}(p)|p\rangle \qquad \hat{\psi}(p) \equiv \langle p|\psi\rangle$$

All of this is related to momentum conservation and we know that it is related to spatial translation invariance. So maybe the key to unlock different descriptions is to look at this kind of problem from the point of view of symmetries.

SPATIAL TRANSLATIONS

Group of translations in 1D : T_1

RULES:

- $T(a_1) * T(a_2) = T(a_1+a_2)$
- $T(0) = e$
- $T(a)^{-1} = T(-a)$

ISOMORPHYSM: $T_1 \cong (\mathbb{R},+)$

We can define a map (homomorphysm, i.e. it preserves the Group structure) between T_1 and $(\mathbb{R},+)$:

$$\Phi : T_1 \longrightarrow (\mathbb{R},+) \qquad \Phi(T(a)) \longmapsto a$$

$$\begin{cases} \Phi(T(a) * T(b)) = \Phi(T(a))\, \Phi(T(b)) \\ \text{neutral element} : 0 \\ \text{inverse element} : -a \end{cases}$$

This gives a way to visualize the transformation. (N.B. in this course we adopt the active point of view of transformations)

Consider the translation : $x \longrightarrow x' = x+a$ (non homogeneus transformation, i.e. we cannot describe it as a matrix multiplication). However there's a simple trick to transform it in an homogeneus one : idea change representation.

We can define the map:

$$\Phi : T_1 \longrightarrow GL(2,\mathbb{R})$$

$$\Phi(T(a)) \longmapsto \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$$

it is a representation : it is an homomorphysm and a bijection. Therefore it gives the opportunity to define T_1 as:

$$T_1 = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in GL(2,\mathbb{R}) \; : \; a \in \mathbb{R} \right\}$$

and now to do the translation we can simply do a matrix multiplication. (i.e. the translation as an homogeneus transformation)

$$\Phi(T(a)) \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x+a \\ 1 \end{pmatrix}$$

Now, therefore, we can immediately figure out what is its algebra.

$$\Sigma = \{ \underset{\substack{\downarrow \\ N\times N \\ \text{complex matrices}}}{x \in M(N,\mathbb{C})} \; : \; e^{tx} \in G,\ t \in \mathbb{R} \}$$

$$\longrightarrow \quad e^{tx} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} \simeq \mathbb{1}_{2\times2} + tx = \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix}$$

So the <u>algebra of the group</u> is simply :

$$t(1) \equiv \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}, \; y \in \mathbb{R} \right\} \qquad t(1) \cong \mathbb{R}$$

<u>Problem</u> : the 2-dim finite representation that we've just found is not unitary. This is due by the fact that the group is not compact.
I would like to have instead a unitary representation because generators of unitary repr. are hermitian objects (the typical op. we're interested in in QM)
(First candidates to be observables).

HOW TO FIND A UNITARY REPRESENTATION

The idea is to <u>consider functions</u> . Let's define :

$$\psi'(x_0) \equiv \psi(x_0 - a) \qquad \rightarrow \qquad \psi'(x) = \psi(x-a)$$

The idea is to interpreter ψ' as :

$$\psi(x) \longrightarrow \psi'(x) = \mathcal{D}(a)\psi(x) \big(= \psi(x-a) \big)$$

$$\psi(x-a) = \psi(x) - a\frac{d}{dx}\psi(x) + \frac{1}{2}a^2\frac{d^2}{dx^2}\psi(x) + \dots = \left(1 - a\frac{d}{dx} + \frac{1}{2}a^2\frac{d^2}{dx^2} + \dots \right)\psi(x)$$

$$\longrightarrow \quad \mathcal{D}(a) = e^{-a\frac{d}{dx}} \equiv e^{-iaP} \qquad P = -i\frac{d}{dx}$$

math convention

Phys convention

exponential map

<u>generator of the algebra</u>

(It is only one because we are in 1 DIM)

<u>OBSERVATION</u> : ⚠
- It is an homomorphism
- It is a bijection

⟶ is a valid representation

But wait, also in the 2x2 finite dimension we can define the exponential map

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \exp\left(-ia \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \right) = e^{-iaP} \qquad P \text{ is a 2x2 matrix}$$

The structure between the Lie group and the Lie algebra is given by the same exponential map. However the generators have a totally different form : in the 1st case is a differential operator in the 2nd case is a 2x2 matrix. Moreover in the 2nd case we can say quickly that the representation is not unitary because the generators are not hermitian.

To check that the 1st representation is unitary we need to equipe the space with an inner product (in order to check unitarity). The space of functions is $L^2(\mathbb{R})$

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} dx \; f^*(x)\, g(x) \longrightarrow \langle f, Pg \rangle = \langle Pf, g \rangle \rightarrow \text{the generator is hermitian.} \rightarrow \text{the representation is unitary.}$$

FOURIER SPACE POINT OF VIEW

The fact that translations in this representation act as derivative in space strongly suggest to look at this description from the point of view of Fourier space : in which derivatives act as products !

i) Inverse Fourier transform: $\psi(x) = \int_{-\infty}^{+\infty} dp\, e^{-ipx}\, \hat{\psi}(p)$

ii) Action of translation in Fourier space : $\hat{\psi}'(p) = e^{-iap}\hat{\psi}(p)$

iii) Action of translation on Ket (vector space) ?

Given that $\hat{\psi}(p) \equiv \langle p|\psi\rangle$, what is $|\psi\rangle \longrightarrow |\psi'\rangle$?

It's possible to show that $|\psi\rangle \longrightarrow |\psi'\rangle = e^{-iaP}|\psi\rangle$ if $P|p\rangle = p|p\rangle$

So we learn that we have a representation acting on momentum eigenstates :

$$\mathcal{D}(a)|p\rangle = e^{-iaP}|p\rangle = e^{-iap}|p\rangle \qquad (\text{where } P|p\rangle = p|p\rangle)$$

Consider the sub-space given by just 1 single vector $\{|p\rangle\}$ with p fixed. Therefore the representation that we just wrote is irreducible , because $\forall a$ P remains the same.
However it is not unitary !

Generelization in n - dimension

In this case the group is n-dimensional , since the element of the group is defined by $\vec{a}$ that has n components. and it is isomorphic to $\mathbb{R}^n$. We can extract the algebra just using the definition :

$$\left\{ X = \left(\begin{array}{c|c} 0_{n\times n} & \vec{y} \\ \hline 0\cdots 0 & 0 \end{array}\right) \; : \; \vec{y} \in \mathbb{R}^n \right\}$$

The generators , which form a basis in the Lie algebra , are :

$$\left\{ T^1 = \left(\begin{array}{c|c} 0_{n\times n} & \begin{matrix}1\\0\\\vdots\\0\end{matrix} \\ \hline 0\ldots 0 & 0 \end{array}\right) , \ldots , T^n = \left(\begin{array}{c|c} 0_{n\times n} & \begin{matrix}0\\0\\\vdots\\1\end{matrix} \\ \hline 0\ldots 0 & 0 \end{array}\right) \right\}$$

We can see that these generators commute with each other. $[T^i, T^j] = 0$

We can take the exponential map an find the unitary representation :

$$\mathcal{D}(\vec{a}) = e^{\vec{a}\cdot\vec{T}} = e^{-i\vec{a}\cdot(i\vec{T})} \qquad \vec{P} = i\vec{T}$$

Therefore the equation $P|p\rangle = p|p\rangle$ is replaced by $\mathcal{D}(\vec{a})|p\rangle = e^{-i\vec{a}\cdot\vec{P}}|p\rangle$

So the endpoint is that we can describe a non relativistic particle in Q.M. according to unitary representations of the symmetry group.
Therefore, maybe, in order to describe a relativistic particle we should find a unitary representation of the Poincare group which maybe it is quite similar to the one that we saw. However we'll have to consider also the presence of boost, and rotations.

POINCARÉ GROUP

Lecture 2 27/02/2024

Def (POINCARÉ GROUP) : The Poincare group is a 10-dimensional non compact and non-abelian Lie group of Minkowski spacetime isometries

↓ $\mathcal{M} = \mathbb{R}^4$ equipped with a Lorentzian metric $g = \mathrm{diag}(1,-1,-1,-1)$

↓ the interval between events is left invariant $\Delta S^2 = (t-t_0)^2 - |\vec{x}-\vec{x}_0|^2$

We can have 2 Kind of transformations :

- 4-DIMENSIONAL SPACE-TIME TRANSLATIONS : they rigidly translate in space-time the points of MinKowski space. These transformations form the group $T_{1,3}$.
- TRANSFORMATIONS THAT LEAVE INVARIANT ΔS^2 : (ΔS^2 is the distance from the origin in $\mathcal{M}$) These transformations form the so called Lorentz Group

LORENTZ GROUP

Considering the metric $g = \mathrm{diag}(1,-1,-1,-1)$ and taking a 4-vector $\begin{pmatrix} t \\ \vec{x} \end{pmatrix}$ we can see that:

$$(t, \vec{x}^T)\, g \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = (t, -\vec{x}^T) \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = t^2 - |\vec{x}|^2$$

Consequently if we take a transformation Λ

$$\begin{pmatrix} t \\ \vec{x} \end{pmatrix} \longrightarrow \Lambda \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$$

it leaves ΔS^2 invariant iff:

$$(t, \vec{x}^T)\, \Lambda^T g \Lambda \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = (t, \vec{x}^T)\, g \begin{pmatrix} t \\ \vec{x} \end{pmatrix} \longrightarrow \boxed{\Lambda^T g \Lambda = g}$$

Therefore we can mathematically define the Lorentz group as:

$$\boxed{O(1,3) = \{ \Lambda \in GL(4, \mathbb{R}) \mid \Lambda^T g \Lambda = g \} \ \underline{\text{with}} \ \ g = \mathrm{diag}(1,-1,-1,-1)}$$

We note that the condition $\Lambda^T g \Lambda = g$ can be also written in the form:

$$\boxed{\Lambda^T = g \Lambda^{-1} g^{-1}}$$

It's simply to prove tha $O(1,3)$ is a group i.e. it satisfies: closure $\Lambda_1 \Lambda_2 \in O(1,3)$ has an identity element $\mathbb{1}_{4\times 4}$, it has an inverse element Λ^{-1} such that $\Lambda \Lambda^{-1} = \mathbb{1}$.

Comment

Using the language of 4 vectors (tensorial language) we could rewrite $\Lambda^T g \Lambda = g$ in a tensorial way. We define:

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad \delta^\mu{}_\nu = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

We collect space-time coordinates into the 4-vector with upper (contravariant) components:

$$x^\mu = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix} \longrightarrow \Delta s^2 = t^2 - |\vec{x}|^2 = x^\mu x^\nu g_{\mu\nu}$$

If we introduce the covariant version $x_\mu = x^\rho g_{\rho\mu}$ (where we say that indices are lowered using the metric tensor $g_{\mu\nu}$) we have that $t^2 - |\vec{x}|^2 = x^\mu x_\mu$

Contrariwise, indices are raised by the inverse metric tensor:

$$(g^{-1})^{\mu\nu} \text{ soch that } (g^{-1})^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \longrightarrow (g^{-1})^\mu{}_\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

In other words in flat MinKowski space $g = g^T = g^{-1}$. Typically people just write $g_{\mu\nu} = g^{\mu\nu}$

$$x_\mu = x^\rho g_{\rho\mu} \longrightarrow (g^{-1})^{\mu\sigma} x_\mu = x^\rho g_{\rho\mu} (g^{-1})^{\mu\sigma} = x^\rho \delta^\sigma_\rho = x^\sigma$$

The action of the Lorentz group can be written as:

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Λ leaves invariant $x^2 = x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu$. There fore:

$$x^2 \longrightarrow x'^2 = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho x^\sigma = x^2 = g_{\rho\sigma} x^\rho x^\sigma$$

$$\longrightarrow \boxed{\Lambda^T g \Lambda = g} \longleftrightarrow \boxed{g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}}$$

OBSERVATIONS

i) $\Lambda^\mu{}_\nu$ defines the L.T. acting on a contravariant 4-vector $x'^\mu = \Lambda^\mu{}_\nu X^\nu$

ii) Consider now $\Lambda_{\mu\nu} = g_{\mu\rho} \Lambda^\rho{}_\nu$. We can write: $(\Lambda^T)_{\mu\nu} = \Lambda_{\nu\mu}$. Consequently we have

$$\boxed{(\Lambda^T)_\mu{}^\nu = \Lambda^\nu{}_\mu}$$

iii) We can rewrite $g^{-1} \Lambda^T g = \Lambda^{-1}$ as

$$(\Lambda^{-1})^\mu{}_\nu = (g^{-1} \Lambda^T g)^\mu{}_\nu = (g^{-1})^{\mu\rho} (\Lambda^T)_\rho{}^\sigma g_{\sigma\nu} = (g^{-1})^{\mu\rho} \Lambda^\sigma{}_\rho \, g_{\sigma\nu} = \Lambda_\nu{}^\mu \longrightarrow \boxed{(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu}$$

iv) Manipulating $x^\mu = \Lambda^\mu{}_\nu x^\nu$ we can find that, for covariant 4-vectors: $x'_\mu = \Lambda_\mu{}^\nu x_\nu$. that can be rewritten as:

$$\boxed{x'_\mu = \Lambda_\mu{}^\nu x_\nu = (\Lambda^{-1})^\nu{}_\mu x_\nu = x_\nu (\Lambda^{-1})^\nu{}_\mu}.$$

If we interpret $x_\mu = (t, -x^T)$ then $(t, -x^T) \longrightarrow (t, -x^T)\Lambda^{-1}$

From the condition $\Lambda^T g \Lambda = g$ we note that: $\det(\Lambda^T g \Lambda) = \det(g) \longrightarrow \det(\Lambda)^2 \cancel{\det(g)} = \cancel{\det(g)}$

$$\longrightarrow \det(\Lambda)^2 = 1 \longrightarrow \boxed{\det(\Lambda) = \pm 1}$$

So it makes sense to define:

$$\boxed{SO(1,3) = \{\Lambda \in O(1,3) \mid \det(\Lambda) = +1\}} \qquad \boxed{O_-(1,3) = \{\Lambda \in O(1,3) \mid \det(\Lambda) = -1\}}$$

Moreover we note that:

$$g_{00} = +1 = g_{\mu\rho}\Lambda^\mu{}_0\Lambda^\rho{}_0 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3}(\Lambda^i{}_0)^2 \longrightarrow (\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3}(\Lambda^i{}_0)^2 \geq 1$$

$$\longrightarrow \boxed{\overset{\uparrow}{\Lambda^0{}_0 \geq 1} \quad \vee \quad \overset{\downarrow}{\Lambda^0{}_0 \leq -1}}$$

So in total we can have 4 subgroups

only this is a group!
(It is the only to contain the neutral element)

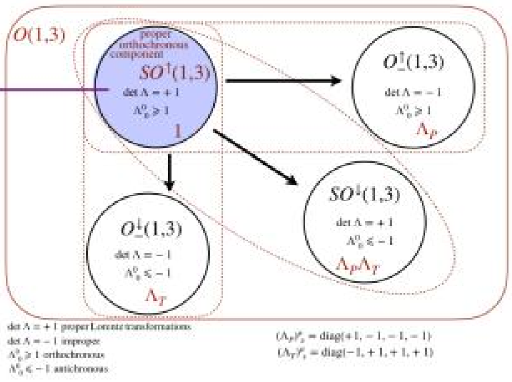


Figura 1.3: *Structure of the Lorentz group.*

$$\rightarrow O(1,3) = \underbrace{SO^\uparrow(1,3) \cup SO^\downarrow(1,3)}_{SO(1,3)} \cup \underbrace{O_-^\uparrow(1,3) \cup O_-^\downarrow(1,3)}_{O_-(1,3)}$$

It is always possible to reduce a generic Lorentz transformation to a restricted one; introducing the three operators

$$\underset{O_-^\downarrow(1,3)}{\underset{\cap}{\Lambda_T = \text{diag}(-1,1,1,1)}} \qquad \underset{O_-^\uparrow(1,3)}{\underset{\cap}{\Lambda_P = \text{diag}(1,-1,-1,-1)}} \qquad \underset{SO^\downarrow(1,3)}{\underset{\cap}{\Lambda_P\Lambda_T = \text{diag}(-1,-1,-1,-1)}}$$

1) Consider $\Lambda \in O_-^\uparrow(1,3) \longrightarrow \Lambda_P\Lambda \equiv \tilde{\Lambda} \in SO^\uparrow(1,3)$ and since $\Lambda_P^{-1} = \Lambda_P \rightarrow \Lambda = \Lambda_P\tilde{\Lambda}$

2) Consider $\Lambda \in O_-^\downarrow(1,3) \longrightarrow \Lambda_T\Lambda \equiv \tilde{\Lambda} \in SO^\uparrow(1,3)$ and since $\Lambda_T^{-1} = \Lambda_T \rightarrow \Lambda = \Lambda_T\tilde{\Lambda}$

3) Consider $\Lambda \in SO^\downarrow(1,3) \longrightarrow \Lambda_P\Lambda_T\Lambda \equiv \tilde{\Lambda} \in SO^\uparrow(1,3)$ and $\Lambda = \Lambda_P\Lambda_T\tilde{\Lambda}$

PARAMETERS OF THE LORENTZ GROUP

The number of indipendent real parameters is 6 : (3 rotations) and (3 boosts).
More formally O(1,3) consists of 4×4 real matrices. So in general there are 16 elements
However the constraint $\Lambda^T g \Lambda = g$ decreases this number from 16 to 6.

LIE ALGEBRA

We can apply the definition we're looking for X such that

$$e^{tX} = \Lambda \quad \text{with} \quad g^{-1}\Lambda^T g = \Lambda^{-1}$$

Consequently :

$$\longrightarrow \quad g^{-1}(e^{tX})^T g = (e^{tX})^{-1}$$

$$g^{-1}\, e^{tX^T} g = e^{-tX}$$

$$e^{t g^{-1} X^T g} = e^{-tX}$$

$$\longrightarrow \quad g^{-1}X^T g = -X \quad \longrightarrow \quad \boxed{X^T g = -gX}$$

$$\longrightarrow \quad \boxed{so(1,3) = \{ X \in M(4,\mathbb{R}) \mid X^T g = -gX \}}$$

Since $g = g^T \longrightarrow \boxed{(Xg)^T = -gX}$. This allow us to rewrite the algebra in a different way:

Consider a generic matrix $X \equiv \begin{pmatrix} A & \vec{b} \\ \vec{c}^T & D \end{pmatrix}$ we write :

$$-gX = -\begin{pmatrix} 1 & 0 \\ 0^T & -\mathbb{1} \end{pmatrix}\begin{pmatrix} A & \vec{b} \\ \vec{c}^T & D \end{pmatrix} = \begin{pmatrix} A & \vec{b} \\ -\vec{c}^T & -D \end{pmatrix} \overset{!}{=} -(gX)^T = \begin{pmatrix} -A & \vec{c} \\ -\vec{b}^T & D^T \end{pmatrix}$$

From the comparison we extract that: $A = 0$; $b = c$; $D = -D^T$:

$$\boxed{so(1,3) = \left\{ X = \begin{pmatrix} 0 & b \\ b^T & D \end{pmatrix} \in M(4,\mathbb{R}) \mid b \in \mathbb{R}^3 \wedge D = -D^T \right\}}$$

A generic element of the algebra takes the form:

$$X = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ K_1 & 0 & a & b \\ K_2 & -a & 0 & c \\ K_3 & -b & -c & 0 \end{pmatrix} \in so(1,3)$$

Then we can write immediatily a basis (generators) :

$$T^1_{so(1,3)} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad T^2_{so(1,3)} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \quad T^3_{so(1,3)} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$T^4_{so(1,3)} = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad T^5_{so(1,3)} = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad T^6_{so(1,3)} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

N.B. T^1, T^2, T^3 is a basis of a subalgebra that is isomorphic to $su(2)$

In the *physicist convention*:

$$(J^1_{4vec})^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{array}\right) \quad (J^2_{4vec})^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{array}\right) \quad (J^1_{4vec})^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad [J^i_{4vec}, J^j_{4vec}] = i\,\varepsilon_{ijk}\,J^k_{4vec}$$

Lorentz transformations generated by linear combinations of J^k_{4vec} describe rotations
N.B. if we look to the entire L.G. the representation is irreducible since it is not possible to find an invariant subspace.; if we restrict to rotations the defining representation is reducible. This is evident by the block-diagonal structure of J_{4vec} : it is not possible to mix temporal and spatial components of a 4 vector. This implies that with a rotation the temporal component is untouched while the spatial part transforms according to repr. of SO(3 acting on vectors of $\mathbb{R}^3$.

The last 3 generators are:

$$(K^1_{4vec})^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & i & 0 & 0 \\ \hline i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (K^1_{4vec})^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & 0 & i & 0 \\ \hline 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad (K^1_{4vec})^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & 0 & 0 & i \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{array}\right)$$

$$[K^i_{4vec}, K^j_{4vec}] = -i\,\varepsilon_{ijk}\,J^k_{4vec}$$

$$[J^j_{4vec}, K^i_{4vec}] = i\,\varepsilon_{ijk}\,K^k_{4vec}$$

Lorentz transformations generated by linear combinations of K^k_{4vec} describe boosts
Contrary to rotations boosts do not form a subgroup of L.G. → if we combine two boosts it is not guaranteed that we obtain a boost (unless if we consider boost on the same dir)

INFINITESIMAL LORENTZ TRANSFORMATIONS (i.e. close to the identity)

$$\Lambda(d\vec{\theta}, d\vec{\eta}) = \mathbb{1}_{4\times4} \underbrace{-i\,d\vec{\theta}\cdot\vec{J}_{4vec} - i\,d\vec{\eta}\cdot\vec{K}_{4vec}}_{\in\, so(1,3)} \in SO^{\uparrow}(1,3)$$

$$\underset{\text{infinitesimal real parameters}}{\uparrow\ \uparrow}$$

It can be rewritten in a more compact tensorial notation considering the anti-symmetric rank-2 tensor $J^{\mu\nu}_{4vec} = -J^{\nu\mu}_{4vec}$ (so it has 6 ind. comp. $J^{01}, J^{02}, J^{03}, J^{12}, J^{23}, J^{23}$)

$$\longrightarrow \boxed{\Lambda(\omega) = \mathbb{1}_{4\times4} - \frac{i}{2}\,\omega_{\mu\nu}\,J^{\mu\nu}_{4vec}}$$

FINITE LORENTZ TRANSFORMATIONS

To move to finite transformation. We consider the exponential map:

$$\begin{aligned} \exp :\ so(1,3) &\longrightarrow SO^{\uparrow}(1,3) \\ X &\longmapsto e^{X} \end{aligned}$$

Well defined : we know that $e^{tX} \in SO^{\uparrow}(1,3)$ → also for $t=1$.

It is possible to show that the exponential map is surjective : any $\Lambda \in SO^{\uparrow}(1,3)$ is the exponential of some element of the Lie algebra of the L.G.

$$\boxed{\Lambda(\vec\theta,\vec\eta) = \exp\left(-i\vec\theta\cdot\vec J_{4vec} - i\vec\eta\cdot\vec K_{4vec}\right)}$$

- ROTATIONS $\boxed{\Lambda_R(\theta,\hat n) = \exp\left(-i\theta\hat n\cdot\vec J_{4vec}\right)}$
- BOOSTS $\boxed{\Lambda_B(\hat p,\eta) = \exp\left(-i\eta\hat p\cdot\vec K_{4vec}\right)}$

SPACE-TIME TRANSLATIONS

$x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu$

These transformations form a group of dim. 4 (4 parameters) called $T_{1,3}$
The generators in this case are $P^\mu = (P^0, P^1, P^2, P^3)$, therefore the finite space-time translation is given by:

$$T(a^\mu) = e^{ia_\mu P^\mu} = e^{ia_0P^0 - i\vec a\cdot\vec P}$$ (see the n-dim. case 🔗)

STRUCTURE OF POINCARE' GROUP

- We identify an element of the Poincare' group as $P(\Lambda,a)$.
- The group is characterized by 10 parameters $\nearrow a^\mu$ $\searrow \vec\theta,\vec\eta$
- Properties:

> ▶ Group composition rule: $P(\Lambda_2,a_2) * P(\Lambda_1,a_1) = P(\Lambda_2\Lambda_1, \Lambda_2a_1 + a_2)$
> ▶ Neutral element: $P(\mathbb{1},0) = e$
> ▶ Inverse element $P(\Lambda,a)^{-1} = P(\Lambda^{-1}, -\Lambda^{-1}a)$

▶ Consider $h = P(\mathbb{1},a)$ (pure space-time transl.) and $g = P(\Lambda,b)$ (generic Poincare' transf.)

$$\longrightarrow \quad g*h*g^{-1} = P(\Lambda,b)*P(\mathbb{1},a)*P(\Lambda^{-1},-\Lambda^{-1}b) = P(\Lambda,b)*P(\Lambda^{-1},-\Lambda^{-1}b+a) = P(\mathbb{1},\Lambda a)$$

It's still a space time translation $\longrightarrow$ $T_{1,3}$ is an invariant subgroup of $SO^\uparrow(1,3)$

▶ Any Poincare' transformation can be written as a product of a pure translation and an homogeneous Lorentz transformation:

$$P(\Lambda,a) = P(\mathbb{1},a) * P(\Lambda,0)$$ ★

In other words the Poincare' group is the semidirect product of the L.G. and the group of space-time translations.

N.B. If we write $P(\Lambda,0) * P(\mathbb{1},a) \notin P$ (do not commute)

Typically people indicate : $ISO^\uparrow(1,3) = T_{1,3} \times SO^\uparrow(1,3)$ (Proper Orthochronus Poincare' group)

Let's consider a generic representation of the full Poincaré group:

$$\mathcal{D}(P(\Lambda,a)) \equiv \mathcal{D}(\Lambda,a)$$

The factorization property ★ is useful to write

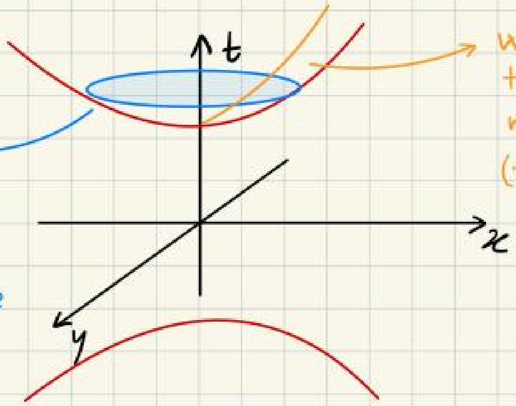
$$\mathcal{D}(\Lambda,a) = \mathcal{D}(a)\cdot\mathcal{D}(\Lambda) = e^{ia_\mu P^\mu} e^{-i\vec{\theta}\cdot\vec{J}-i\vec{\eta}\cdot\vec{K}} \quad ★$$

There are *different ways to parametrize a transformation*. In this course we'll use:

- Parametrization for the rotation: "Axis angle parametrization" $e^{-i\vec{\theta}\cdot\vec{J}} = e^{-i\theta\hat{n}\cdot\vec{J}}$
  Note that these rotations, since the parameters are not bounded, are compact!
- Parametrization for the boost: $e^{-i\eta\hat{p}\cdot\vec{K}}$
  Contrary to rotations η does not live in a bounded interval. In fact

$$\sinh(\eta) = \frac{v}{\sqrt{1-v^2}} \longrightarrow -1<v<+1$$

This fact can be visualized with the Manifold structure of the Lorentz group.



when we rotate we are describing this kind of orbits (which are compact)

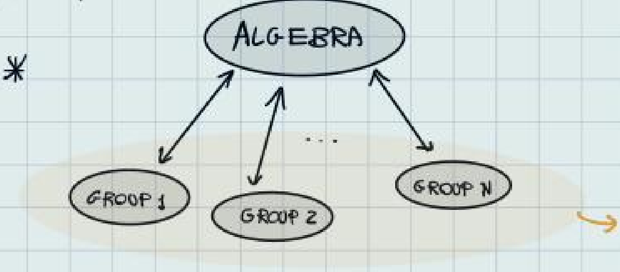
(they leave untouched the time component)

when we boost we describe this kind of orbit (which is not compact)
(they mix the time and the spatial component)

COMMUTATION RELATIONS BETWEEN GENERATORS OF L.G. AND $T_{1,3}$

STRATEGY : representation of the algebra ⟶ representation of the group.
by exponentiating

In general if we start from some algebra then it's possible to have different groups associated to it:



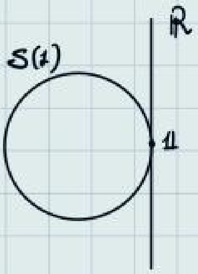
these groups are locally isomorphyc (same algebra)

Consequently if we start from a representation of the algebra is not guaranteed that we get (by exponentiating) a representation for each one of these groups.
It is, instead, garanteed only for the group that is simply connected (it can be only one (among those). This is important to understand the existence of fermions.

Example : U(1) (Circle group)

We Know, in general that : $U(N) = \{ \underset{\substack{\downarrow \\ \text{space of } N\times N \\ \text{matrices with} \\ \mathbb{C} \text{ elements}}}{U \in GL(N,\mathbb{C})} : \underset{\substack{\downarrow \\ \text{unitary}}}{U^{+}U = \mathbb{1}_{N\times N}} \} \xrightarrow{1D} U(1) = \{ z \in \mathbb{C} : |z|^2 = 1 \}$

The manifold of U(1) is just $S(1)$. The algebra instead is the tangent space to the identity : i.e. in this case is $\mathbb{R}$



Also T_1 has $\mathbb{R}$ as algebra . So we find an example of this ✱ situation : U(1) and T_1 share the same algebra , they are locally the same but globally different , in fact for U(1) the manifold is compact while for T_1 the manifold is $\mathbb{R}$ itself.
Therefore if we take a representation of the algebra and we exponentiate it is not garanteed that we get a representation of each group.

For T_1 we found that: $P|p\rangle = p|p\rangle \xrightarrow{\text{exponentiate}} e^{-iaP}|p\rangle = e^{-iap}|p\rangle$ $\boxed{p \in \mathbb{R}}$

Is $\mathbb{R}$ also a representation for U(1)? Of course not!

Because a representation for U(1) is

$$\begin{cases} R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2) \\ R(\theta=0) = e \\ R(\theta)^{-1} = R(-\theta) \end{cases} : \quad R(\theta+2\pi) = R(\theta) \rightarrow e^{ip\cdot\theta} = e^{i(p+2\pi)\cdot\theta} \longleftrightarrow \boxed{p \in \mathbb{Z}}$$

It's useful to write ★ in infinitesimal form, in order to make considerations about the algebra of the group:

$$D(\mathbb{1}+\omega,\varepsilon) \simeq \mathbb{1} + i\,\varepsilon_\mu P^\mu - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}$$

Let's combine 3 Poincaré transformations

$$D(\Lambda,a)\underbrace{D(\mathbb{1}+\omega,\varepsilon)}_{\text{infinitesimal}}D(\Lambda,a)^{-1}$$

we can write this expression in 2 equivalent ways:

1) $\blacktriangleright = D(\Lambda,a)\left(\mathbb{1} + i\varepsilon_\mu P^\mu - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)D(\Lambda,a)^{-1}$

2) $\blacktriangleright = D(\Lambda,a)\,D(\mathbb{1}+\omega,\varepsilon)\,D(\Lambda^{-1},-\Lambda^{-1}a)$
$= D(\Lambda,a)\cdot D(\Lambda^{-1}+\omega\Lambda^{-1}, -\Lambda^{-1}a - \omega\Lambda^{-1}a + \varepsilon)$
$= D(\mathbb{1} + \Lambda\omega\Lambda^{-1}, \cancel{-a} - \Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon + \cancel{a})$ ⟶ this is still an infinitesimal transformation -a and a cancel out.
$= \mathbb{1} - \frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu} + i(-\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)_\mu P^\mu$

Comparing the 2 results, we get <u>2 relations</u>:

- <u>Comparing the P^μ part</u>:

$$D(\Lambda,a)\, P^\rho D(\Lambda,a)^{-1} = \Lambda_\mu{}^\rho P^\mu = (\Lambda^{-1})^\rho{}_\mu P^\mu$$

$$1) \longrightarrow \boxed{D(\Lambda,a)^{-1}\, P^\mu\, D(\Lambda,a) = \Lambda^\mu{}_\nu P^\nu}$$

P^μ transforms as a 4-vector under a Poincare transformation

- <u>Comparing the $J^{\mu\nu}$ part</u>:

$$D(\Lambda,a)\left(-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)D(\Lambda,a)^{-1} = -\frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu} + i\,(-\Lambda\omega\Lambda^{-1}a)_\mu P^\mu$$

$$\longrightarrow D(\Lambda,a)\, J^{\mu\nu} D(\Lambda,a)^{-1} = (\Lambda^{-1})^\mu{}_\rho (\Lambda^{-1})^\nu{}_\sigma (a^\sigma P^\rho - a^\rho P^\sigma + J^{\rho\sigma})$$

[For details: 🔗]

$$\text{If } a^\rho = 0 \quad \longrightarrow D(\Lambda)\, J^{\mu\nu} D(\Lambda)^{-1} = (\Lambda^{-1})^\mu{}_\rho (\Lambda^{-1})^\nu{}_\sigma J^{\rho\sigma} \qquad D(\Lambda,0) \equiv D(\Lambda)$$

$$2) \longrightarrow \boxed{D(\Lambda)^{-1} J^{\mu\nu} D(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma J^{\rho\sigma}}$$

$J^{\mu\nu}$ transforms as a rank-2 tensor under a Poincare transformation

Now we can use the previous two relations to extract the Lie algebra of the Poincare group. We start from (1) and rewrite it as:

$$D(\Lambda,a)\, P^\mu D(\Lambda,a)^{-1} = D(\Lambda,a)\, P^\mu D(\Lambda^{-1}, -\Lambda^{-1}a) = (\Lambda^{-1})^\mu{}_\rho P^\rho$$

We now rewrite it as an infenitesimal transformation: (remember that at 1st order $\Lambda^{-1}a = \varepsilon$)

$$D(\mathbb{1}+\omega,\varepsilon)\, P^\mu D(\mathbb{1}-\omega,-\varepsilon) = \left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} + i\varepsilon_\rho P^\rho\right) P^\mu \left(\mathbb{1} + \frac{i}{2}\omega_{\eta\lambda}J^{\eta\lambda} - i\varepsilon_\eta P^\eta\right) \overset{!}{=} P^\mu - \omega^\mu{}_\rho P^\rho$$

$$\longrightarrow \quad P^\mu - i\varepsilon_\rho (P^\mu P^\rho - P^\rho P^\mu) - \frac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma}P^\mu - P^\mu J^{\rho\sigma}) = P^\mu - \omega^\mu{}_\rho P^\rho$$

$$\longrightarrow \quad P^\mu - i\varepsilon_\rho [P^\mu, P^\rho] - \frac{i}{2}\omega_{\rho\sigma}[J^{\rho\sigma}, P^\mu] = P^\mu - \omega^\mu{}_\rho P^\rho$$

$$\longrightarrow \quad \boxed{[P^\mu, P^\rho] = 0}$$

$$\longrightarrow \quad \frac{i}{2}\omega_{\rho\sigma}[J^{\rho\sigma}, P^\mu] = \omega^\mu{}_\rho P^\rho = g^{\mu\rho}\omega_{\rho\sigma}P^\sigma$$

$$\longrightarrow \quad [J^{\rho\sigma}, P^\mu] = -2ig^{\mu\rho}P^\sigma = -i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho)$$

$$\xrightarrow{\substack{(\text{antisimmetry})\\ \rho\leftrightarrow\sigma}} \quad \boxed{[J^{\rho\sigma}, P^\mu] = i(P^\rho g^{\sigma\mu} - P^\sigma g^{\rho\mu})}$$

Repeating the same for the 2nd relation we get:

$$\boxed{[J^{\rho\sigma}, J^{\mu\nu}] = i(J^{\rho\nu}g^{\sigma\mu} + J^{\mu\rho}g^{\nu\sigma} - J^{\mu\sigma}g^{\nu\rho} - J^{\sigma\nu}g^{\rho\mu})}$$

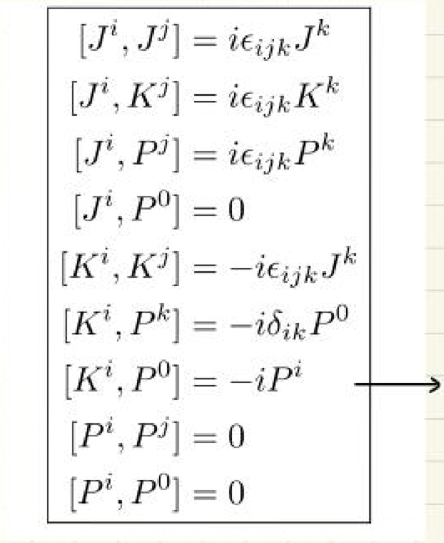
The Lie algebra is constitued by those three relations.

In the three dimensional notation we can rewrite them in a series of commotators:
[see 🔗 for more details]

translation generators: $P^i = \begin{pmatrix} P^1 \\ P^2 \\ P^3 \end{pmatrix}$; $H = P^0$

rotation generators: $J^i = \frac{1}{2}\varepsilon_{ijk} J^{jk} = \begin{pmatrix} J^{23} \\ J^{31} \\ J^{12} \end{pmatrix}$

boost generators: $K^i = \begin{pmatrix} K^1 \\ K^2 \\ K^3 \end{pmatrix} = \begin{pmatrix} J^{01} \\ J^{02} \\ J^{03} \end{pmatrix}$



$$[J^i, J^j] = i\epsilon_{ijk}J^k$$
$$[J^i, K^j] = i\epsilon_{ijk}K^k$$
$$[J^i, P^j] = i\epsilon_{ijk}P^k$$
$$[J^i, P^0] = 0$$
$$[K^i, K^j] = -i\epsilon_{ijk}J^k$$
$$[K^i, P^k] = -i\delta_{ik}P^0$$
$$[K^i, P^0] = -iP^i \rightarrow$$
$$[P^i, P^j] = 0$$
$$[P^i, P^0] = 0$$

boost generators and H do not commute, i.e. they cannot be diagonalized togheter.

CASIMIR OPERATORS of the Poincaré group

After deriving the algebra we are now able to construct the unitary representation of the Poincaré group. First of all is important to introduce the Casimir operators

Casimir operator: special operator that commutes with all the element of the Lie algebra.

The Poincaré group admits 2 Casimir operators:

$$\boxed{P^2 = P_\mu P^\mu}$$

$$\boxed{W^2 = W_\mu W^\mu} \quad \text{with} \quad W^\mu = -\frac{1}{2}\underset{\substack{\downarrow \\ \text{Levi Civita} \\ \text{tensor } \varepsilon^{0123}=1}}{\varepsilon^{\mu\nu\lambda\rho}} J_{\nu\lambda} P_\rho \quad \text{(Pauli Lubanski vector)}$$

CONSTRUCTION OF THE UNITARY REPRESENTATION OF THE POINCARÉ GROUP

The method to construct the unitary representation of the Poincaré group is called method of induced representation. Since we're dealing with a Lie group the logic we'll follow is the following: to construct the elements of the group in a given representation we first focus on the algebra and then we exponentiate. So first we find a representation for the generators, then thanks to the exp. map we find a representation for the elements of the group (that is exactly what we did before with the defining repr. and that's what we'll do now.
So we focus first on the generators. The strategy we'll follow is similar to the one encountered in Q.M. when constructing a "CSCO" (Complete Set of Commuting Operators"). We find such operators we diagonalize them and so we construct the vector space. This strategy is articulated in various steps.

STEP 1: Definition of 4-momentum eigenstates

We start from $T_{1,3}$. The corresponding generators are P^μ. We know that:
$[P^\mu, P^\nu] = 0 \rightarrow P^\mu$ and P^ν can be diagonalized together. The eigenvalue equation is:

$$P^\mu |p^\mu\rangle = p^\mu |p^\mu\rangle$$

We know how these states transform under translation however we know nothing how they transform under $SO^\uparrow(1,3)$ transformations. → we need to add more labels.

STEP 2 : Introduction of the Casimir P^2 and orbits of Poincaré group. (classification)

Since $[P^2, P^\mu]=0$ P^2 and P^μ can be diagonalized together :

$$\begin{cases} P^\mu |p^\mu\rangle = p^\mu |p^\mu\rangle \\ P^2 |p^\mu\rangle = p^2 |p^\mu\rangle \quad \text{with} \quad p^2 = (p^0)^2 - |\vec{p}|^2 \end{cases}$$

Since P^2 is a Casimir, its eigenvalue $p^2=(p^0)^2-|\vec{p}|^2$ remains unchanged under any L.T. we can do a classification based on the value of p^2
4 relevant cases:

it's impossible to go from a place to another

- a) $p^2=0$ $\quad p^0=0, \vec{p}=0$ null vector
- b) $p^2>0$ $\quad$ time-like case
- c) $p^2=0$ $\quad p^0\neq 0, p^0=|\vec{p}|$ light-like case
- d) $p^2<0$ $\quad$ space-like case

- If $p^2>0$ we can show that there is an additional conserved quantity :

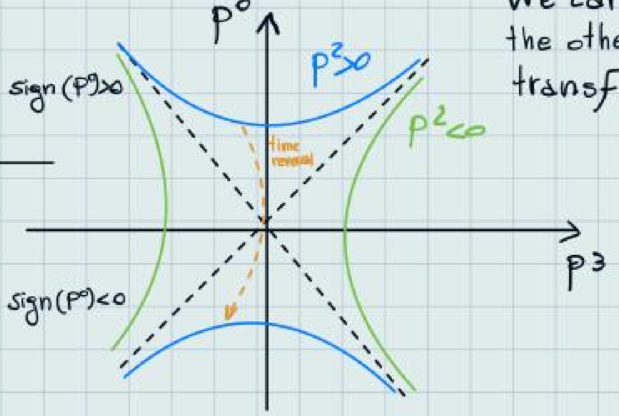
sign (p^0) (Lorentz invariant)

Proof:

Let's consider $p^\mu=(p^0,\vec{p})$; $p^2=(p^0)^2-(p^1)^2-(p^2)^2-(p^3)^2$.

Suppose $p^1=p^2=0 \longrightarrow p^2=(p^0)^2-|p^3|^2 \longrightarrow p_0=\pm\sqrt{(p^2)+(p^3)^2}$

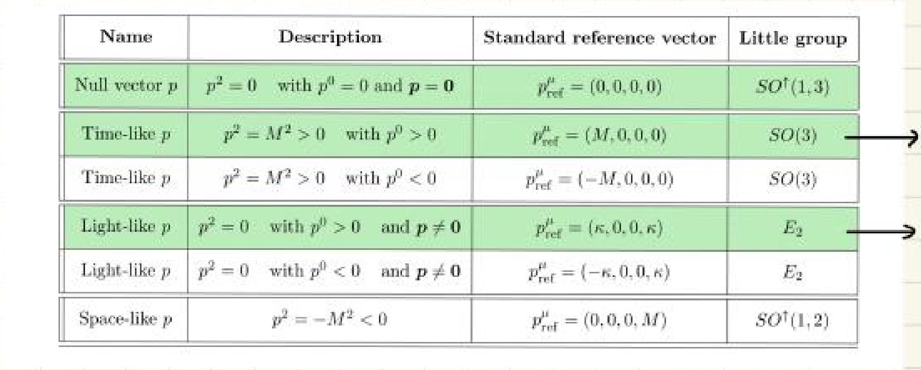
We can move along a branch with a L.T. of $SO^\uparrow(1,3)$



We cannot jump on the others branches with transf. $\in SO^\uparrow(1,3)$

If $p^2>0$ we see that sign p^0 is conserved because only with a time reversal $\notin SO^\uparrow(1,3)$ we can jump to the other branch (contrary to $p^2<0$)

☐



Name	Description	Standard reference vector	Little group
Null vector p	$p^2=0$ with $p^0=0$ and $\boldsymbol{p}=\boldsymbol{0}$	$p^\mu_{\text{ref}}=(0,0,0,0)$	$SO^\uparrow(1,3)$
Time-like p	$p^2=M^2>0$ with $p^0>0$	$p^\mu_{\text{ref}}=(M,0,0,0)$	$SO(3)$
Time-like p	$p^2=M^2>0$ with $p^0<0$	$p^\mu_{\text{ref}}=(-M,0,0,0)$	$SO(3)$
Light-like p	$p^2=0$ with $p^0>0$ and $\boldsymbol{p}\neq\boldsymbol{0}$	$p^\mu_{\text{ref}}=(\kappa,0,0,\kappa)$	E_2
Light-like p	$p^2=0$ with $p^0<0$ and $\boldsymbol{p}\neq\boldsymbol{0}$	$p^\mu_{\text{ref}}=(-\kappa,0,0,\kappa)$	E_2
Space-like p	$p^2=-M^2<0$	$p^\mu_{\text{ref}}=(0,0,0,M)$	$SO^\uparrow(1,2)$

massive case

massless case

MASSIVE CASE

STEP 3 (Massive case $p^2=M^2$ $M>0$) Identification of a reference vector

We rewrite the eigenvalue equation system as:

$$\begin{cases} P^\mu |M,p^\mu\rangle = p^\mu |M,p^\mu\rangle \\ P^2 |M,p^\mu\rangle = M^2 |M,p^\mu\rangle \end{cases}$$

Those equations are valid for each p^μ and M such that $p^2=M^2$, $M>0$. However I make a specific choice: I move to the reference frame where:

$$\text{Reference vector}: \quad P^\mu_{Ref} = (M,\vec{0})$$

$$\longrightarrow \begin{cases} P^\mu |M,\vec{0}\rangle = P^\mu_{Ref} |M,\vec{0}\rangle \\ P^2 |M,\vec{0}\rangle = M^2 |M,\vec{0}\rangle \end{cases}$$

STEP 4: (Massive case) Identification of the Little group

Def (Little group): set of restricted L.T. which leave the reference vector invariant

Given a p^μ (fixed): $G(P) = \left\{ \Lambda \in SO^\uparrow(1,3) : \Lambda^\mu{}_\nu p^\nu = p^\mu \right\}$

we can check that it's actually a group (subgroup)

We can apply this generic definition where $p^\mu = P^\mu_{Ref}$

$$\longrightarrow G(P_{ref}) = \left\{ \Lambda \in SO^\uparrow(1,3) : \Lambda^\mu{}_\nu P^\nu_{Ref} = P^\mu_{Ref} \right\}$$

A possible candidate for our case $P^\mu_{Ref} = (M,\vec{0})$ is $\Lambda_R = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & R \end{array}\right) \rightarrow SO(3)$

$$\longrightarrow \boxed{G(P^\mu_{Ref}) = SO(3)} \quad \text{Little group for the massive case}$$

Parentheses: In this case $G(P)$ is quite trivial. In the massless case $P^2=0$, $p^0>0$, instead, $G(P)$ is more difficult to find. We need a criterium to identify it:

Theorem (Generators of $G(P)$)

The little group $G(P)$ is generated by: $\vec{W} = \vec{J} + \dfrac{\vec{K}\times\vec{P}}{P^0}$ with $P^\mu = (p^0,\vec{p})$

Proof:

A generic $\Lambda \in SO^\uparrow(1,3)$ is defined by 6 parameters and obtained by exponentiating a linear combination of all 6 generators of L.G. Imposing $\Lambda^\mu{}_\nu P^\nu = P^\mu$ (which defines $\Lambda \in G(P)$) imposes a set of constraints that allow only specific combinations of generators to appear, removing 3 d.o.f. It's useful to move in a neighborhood of $\mathbb{1}$:

$$\Lambda \simeq \mathbb{1} - \frac{i}{2}\omega_{\rho\sigma} J^{\rho\sigma}_{4vec}$$

Tensorial notation $\longrightarrow \quad \Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \frac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma}_{4vec})^{\mu}{}_{\nu}$

Observations:

1) Antisymmetry under $\rho \leftrightarrow \sigma \longrightarrow \omega_{\rho\sigma} = -\omega_{\sigma\rho}$

2) Relation between $J^{0i}_{4vec} \leftrightarrow K^{i}_{4vec} \longrightarrow J^{ij}_{4vec} = \varepsilon_{ijk} K^{k}_{4vec}$

It turns out that there is a compact way to write $(J^{\rho\sigma}_{4vec})^{\mu}{}_{\nu}$

$$\longrightarrow (J^{\rho\sigma}_{4vec})^{\mu}{}_{\nu} = i\,(g^{\mu\rho}\delta^{\sigma}{}_{\nu} - g^{\mu\sigma}\delta^{\rho}{}_{\nu})$$

Little Group cond.

$$\boxed{\Lambda^{\mu}{}_{\nu} P^{\nu}_{ref} = P^{\mu}_{ref}} \longrightarrow \quad [\delta^{\mu}{}_{\nu} - \frac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^{\mu}{}_{\nu}]P^{\nu} = P^{\mu}$$

$$\cancel{P^{\mu}} - \frac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^{\mu}{}_{\nu}P^{\nu} = \cancel{P^{\mu}}$$

$$-\frac{i}{2}\omega_{\rho\sigma}\, i\,(g^{\mu\rho}\delta^{\sigma}{}_{\nu} - g^{\mu\sigma}\delta^{\rho}{}_{\nu})P^{\nu} = 0$$

$$\omega_{\rho\sigma} g^{\mu\rho}\delta^{\sigma}{}_{\nu}P^{\nu} = 0$$

$$\omega_{\rho\sigma} g^{\mu\rho}P^{\sigma} = 0$$

$$\boxed{\omega^{\mu}{}_{\sigma}P^{\sigma} = 0} \quad : \text{constraint}$$

In conclusion we have the transformation generated by the linear combination of Lorentz generators $\omega_{\rho\sigma}J^{\rho\sigma}_{4vec}$ with the parameters ω restricted by $\omega^{\mu}{}_{\sigma}P^{\sigma} = 0$.

$$\frac{1}{2}\omega_{\rho\sigma}J^{\rho\sigma}_{4vec} \overset{\text{parametrization}}{=} d\vec{\theta}\cdot\vec{J}_{4vec} + d\vec{\eta}\cdot\vec{K}_{4vec} \qquad d\eta^{i} = \omega^{i0}; \quad d\theta^{k} \equiv \frac{1}{2}\varepsilon_{kij}\,\omega^{ij}$$

$\boxed{\mu = 0}$:

$$\begin{aligned} \omega^{0}{}_{\nu}p^{\nu} &= \cancel{\omega^{0}{}_{0}}p^{0} + \omega^{0}{}_{i}p^{i} \\ &= -\omega^{0i}p^{i} = \\ &= \omega^{i0}p^{i} = \\ &= d\vec{\eta}\cdot\vec{p} = 0 \quad \longrightarrow \quad \boxed{d\vec{\eta}\cdot\vec{p} = 0} \end{aligned}$$

$\boxed{\mu = i}$:

$$\begin{aligned} \omega^{i}{}_{\nu}p^{\nu} &= \omega^{i}{}_{0}p^{0} + \omega^{i}{}_{j}p^{j} = \\ &= \omega^{i}{}_{0}p^{0} - \omega^{ij}p^{j} = \\ &= d\eta^{i}p^{0} - \varepsilon_{ijk}d\theta^{k}p^{j} = \\ &= d\eta^{i}p^{0} + \varepsilon_{ikj}d\theta^{k}p^{j} = 0 \quad \longrightarrow \quad d\vec{\eta}\,p^{0} + d\vec{\theta}\times\vec{p} = 0 \quad \longrightarrow \quad \boxed{d\vec{\eta} = -\frac{d\vec{\theta}\times\vec{p}}{p^{0}}} \end{aligned}$$

Consequently, we find that, of the 6 parameters which define a generic L.T. only 3 are truly independent. We can rewrite ★ as:

$$\frac{1}{2}\omega_{\rho\sigma}J^{\rho\sigma}_{4vec} = d\vec{\theta}\cdot\vec{J}_{4vec} - \frac{d\vec{\theta}\times\vec{p}}{p^{0}}\cdot\vec{K}_{4vec}$$

Using $a\cdot(b\times c) = c\cdot(a\times b) = b\cdot(c\times a)$:

$$\frac{1}{2}\omega_{\rho\sigma}J^{\rho\sigma}_{4vec} = d\vec{\theta}\cdot\vec{J}_{4vec} - \frac{d\vec{\theta}\times\vec{p}}{p^{0}}\cdot\vec{K}_{4vec} = d\vec{\theta}\cdot\vec{J}_{4vec} - d\vec{\theta}\cdot\frac{\vec{p}\times\vec{K}_{4vec}}{p^{0}} =$$

$$= d\vec{\theta} \cdot \left(\vec{J}_{4vec} - \frac{\vec{P} \times \vec{K}_{4vec}}{P^0} \right) = d\vec{\theta} \cdot \left(\vec{J}_{4vec} + \frac{\vec{K}_{4vec} \times \vec{P}}{P^0} \right)$$

Therefore we demostrated that a generic element $\in G(P)$ can be expressed as a linear combination of the 3 generators $\vec{J}_{4vec}$, $\vec{K}_{4vec}$, $\vec{P}$:

$$\vec{W} = \vec{J} + \frac{\vec{K} \times \vec{P}}{P^0} \qquad \forall\, P_{ref} \qquad \square$$

Now, let's take the generators of the little Group $G(P) = SO(3)$: $\vec{J}$ (massive case).
We Know that in general $\vec{J}$ an P^μ do not commute since:

$$[\vec{J}, P^0] = 0 \quad \text{but} \quad [J^i, P^k] = i\varepsilon_{ijk} P^k$$

If we restrict to the subspace that corresponds to the reference vector then we have $\vec{P} = 0$ it is possible to have common diagonalization.
Therefore we can write the following eigensystem:

$$\boxed{\begin{aligned} P^\mu |M, \vec{0}, j, \sigma\rangle &= P^\mu_{ref} |M, \vec{0}, j, \sigma\rangle \\ P^2 |M, \vec{0}, j, \sigma\rangle &= M^2 |M, \vec{0}, j, \sigma\rangle \\ |J|^2 |M, \vec{0}, j, \sigma\rangle &= j(j+1) |M, \vec{0}, j, \sigma\rangle \bigstar \\ J^3 |M, \vec{0}, j, \sigma\rangle &= \sigma |M, \vec{0}, j, \sigma\rangle \end{aligned}}$$

Observations:

i) We recall that our goal is to append additional labels to the states $|M, \vec{0}\rangle$. We constructed the state

$$|M, \vec{0}, j, \sigma\rangle = |M, \vec{0}\rangle \otimes |j, \sigma\rangle$$

the state $|j, \sigma\rangle$ gives the irreducible representation of the rotation group $SO(3)$. Given the last 2 equations we automatically Know how $|M, \vec{0}, j, \sigma\rangle$ transforms under rotations

Consider a pure rotation $\Lambda_R(\hat{n}, \theta) = e^{-i\theta \cdot \hat{n} \cdot \vec{J}_{4vec}}$ (defined by $\theta, \hat{n}$; with Λ_R that acts on 4-vectors). How is the action represented on $|M, \vec{0}, j, \sigma\rangle$?
The rotation will be represented by a certain operator $D(\hat{n}, \theta)$:

$$D(\hat{n}, \theta) = e^{-i\theta \cdot \hat{n} \cdot \vec{J}} \quad \text{such that:}$$

$$D(\hat{n}, \theta) |M, \vec{0}, j, \sigma\rangle = e^{-i\theta \hat{n} \cdot \vec{J}} |M, \vec{0}, j, \sigma\rangle = |M, \vec{0}\rangle \otimes e^{-i\theta \hat{n} \cdot \vec{J}} |j, \sigma\rangle =$$

$$= \sum_\sigma |M, \vec{0}\rangle \otimes |j, \sigma\rangle \langle j, \sigma| e^{-i\theta \hat{n} \cdot \vec{J}} |j, \sigma\rangle$$

I can define:

$$\boxed{\left[D_j(\hat{n}, \theta) \right]_{\sigma\sigma'} =: \langle j, \sigma| e^{-i\theta \hat{n} \cdot \vec{J}} |j, \sigma\rangle = (2j+1)(2j+1)}$$

this is nothing else that the matrix form of the group element in the irreducible representation of the rotation group (labeled by j). We can compute this matrix without problems; infact we Know how J_3 and J_1, J_2 (expressed as comb. of $J_\pm$) act:

$$J^3 |j, \sigma\rangle = \sigma |j, \sigma\rangle \quad ; \quad J^\pm |j, \sigma\rangle = [j(j+1) - m(m\pm1)]^{1/2} |j, \sigma \pm 1\rangle$$

with $J^1 = (J^+ + J^-)/2$ and $J^2 = -i(J^+ + J^-)/2$

$$\longrightarrow \boxed{D(\hat{n},\theta)|M,\vec{0},j,\sigma\rangle = \sum_{\sigma'}\left[D_j(\hat{n},\theta)\right]_{\sigma'\sigma}|M,\vec{0},j,\sigma'\rangle}$$

ii) The operators $\vec{J}$ verify the algebra of angular momentum. In general $\vec{J}$ is the total angular momentum however in our case the $|M,\vec{0},j,\sigma\rangle$ describe particles at rest $\vec{P}_{ref}=0 \longrightarrow \emptyset$ orbital angular momentum.

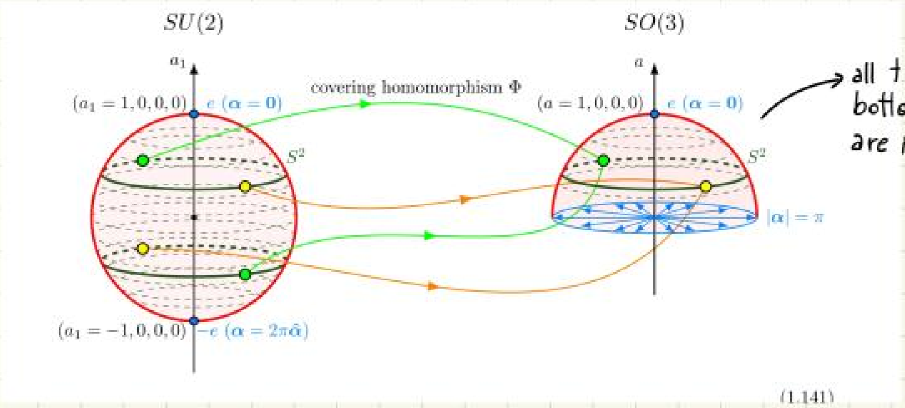
$\longrightarrow$ j describes the spin of the particle $\Rightarrow$ $\boxed{j=s}$

iii) What are the possible values of s? TRICKY POINT
In principle $j=0,\frac{1}{2},1,\frac{3}{2},2,\dots$ and $-j\le\sigma\le+j$

There are 2 groups that share the same algebra of angular momentum : SU(2) and SO(3) (same tangent space at $\mathbb{1}$) → locally the same but globally different.

SU(2) - Manifold : S(3)

SO(3) - Manifold : half of a sphere



all the points at the bottom (antipodal) are identified together

Since we have two groups that share the same algebra, there is only one of them which is topological simply connected : SU(2)

$\longrightarrow$ If we start from a representation of the algebra we get by exponentiating a representation for the group SU(2).

SU(2)	SO(3)
$j=0,1,2,3,\dots$ $j=\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots$	$j=0,1,2,3$

Conventionally the simply connected one is called universal covering i.e. SU(2) is the universal covering of SO(3).

What set of j should I take? Half integers, integers, half integers + integers? Two perspective to answer. First of all note that we do not have too much infos to say that the Little Group is SO(3). The naive argument was that SO(3) left untouched P_{ref}. If we take a generic 4 vector and we decompose it

P_0 does not transform under rotation

$$P = \begin{pmatrix} P_0 \\ \vec{P} \end{pmatrix} = \mathbb{1}\otimes 3$$

the 3-vector transforms under the representation with j=1

Since the representation with $j=1$ is shared by both groups we cannot assign to $G(P)$, $SO(3)$ or $SU(2)$.

1) GROUP THEORY ARGUMENT

From a G.T. perspective $SU(2)$ is more fundamental than $SO(3)$ because it is the universal covering of $SO(3)$ (therefore if we take the repr. of the algebra and we exponentiate it we get the repr. of $SU(2)$

I would be tempted to assign $SU(2)$ as a little group and therefore include also half integer spin

2) Q.M. ARGUMENT : Why $j=\frac{1}{2},\frac{3}{2},\dots$ are not representations of $SO(3)$?

Let's take $j=\frac{1}{2}$: $J^1=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$; $J^2=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$; $J^3=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ (Pauli matricies)

(2D representation of the algebra)

$$-i\alpha_A\cdot J^A=-i\vec{\alpha}\cdot\vec{J}=-i\,|\vec{\alpha}|\,\hat{\alpha}\cdot\vec{J}\xrightarrow{\text{exponentiating}}D_{\frac{1}{2}}(|\alpha|,\hat{\alpha})=\begin{pmatrix}\cos\frac{|\vec{\alpha}|}{2}-i\hat{\alpha}^3\sin\frac{|\alpha|}{2} & -(\hat{\alpha}^2+\hat{\alpha}^1)\sin\frac{|\vec{\alpha}|}{2}\\ (\hat{\alpha}^2-i\hat{\alpha}^1)\sin\frac{|\vec{\alpha}|}{2} & \cos\frac{|\vec{\alpha}|}{2}+i\hat{\alpha}^3\sin\frac{|\vec{\alpha}|}{2}\end{pmatrix}$$

Let's take 2 rotations :

$$D_{\frac{1}{2}}(\theta_1,\hat{\alpha})\,D_{\frac{1}{2}}(\theta_2,\hat{\alpha})=D_{\frac{1}{2}}(\theta_1+\theta_2,\hat{\alpha})$$

and let's take the case in which $\theta_1=\theta_2=\pi$

$$\longrightarrow\quad D_{\frac{1}{2}}(2\pi,\hat{\alpha})=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-\mathbb{1}_{2\times2}$$ $\longrightarrow$ $j=\frac{1}{2}$ is <u>not an homomerphysm</u> $\longrightarrow$ <u>not a representation</u>

However they are almost a representation : they fail only for a phase $\longrightarrow$ they are a projective representation. Now we remember that in a Q.M. setting I don't care about the phase and therefore I'm allowed to consider a projective representation

$$\boxed{\text{Therefore I can use both } j=1,2,3,\dots \text{ and } j=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\dots \quad\text{with}\quad -j\le\sigma\le j}$$

Therefore <u>fermions emerge in a very natural way</u>.

The next step is the generalization to the case where $P\neq0$

This is a problem because J does not commute with P !

The Pauli-Lubanski vector can be written into components as :

$$\boxed{W^0=\vec{J}\cdot\vec{P}\qquad\vec{W}=\vec{J}\cdot P^0+\vec{K}\times\vec{P}}\qquad W=\begin{pmatrix}W^0\\\vec{W}\end{pmatrix}$$

Therefore, applying them to P_{ref} :

$$W^0|M,\vec{0},j,\sigma\rangle=0$$

$$\bigstar\ \vec{W}|M,\vec{0},j,\sigma\rangle=Mj\,|M,\vec{0},j,\sigma\rangle$$

Coseguently, since $W^2 = W^{0 2} - |\vec{W}|^2$ when applied to that state, it gives:

$$★ \quad W^2 |M,\vec{0},j,\sigma\rangle = -M^2 j(j+1)\, |M,\vec{0},j,\sigma\rangle$$

Therefore the idea is that instead of writing ★ I can write ★ . What we gain is that W^2 is a Casimir and therefore this eigenvalue equation remains the same even if we change $P \neq P_{ref}$.

From now on I will replace $j \leftrightarrow s$ because we know that j is the spin:

$$\boxed{W^2 |M,\vec{0},s,\sigma\rangle = -M^2 s(s+1) |M,\vec{0},s,\sigma\rangle}$$

Let's rewrite the eigenvalue equation for J^3 in terms of the W (we can use the 3^{rd} component of ★) because J^3 dos not commute with a generic P $[J^3, P^\nu] \neq 0$ but $[W^3, P^\nu] = 0$

$$\longrightarrow \quad W^3 |M,\vec{0},S,\sigma\rangle = MJ^3 |M,0,s,\sigma\rangle = M\sigma\, |M,\vec{0},s,\sigma\rangle$$

$$\longrightarrow \quad \boxed{\frac{W^3}{M} |M,\vec{0},S,\sigma\rangle = \sigma\, |M,\vec{0},S,\sigma\rangle}$$

That we can rewrite defining

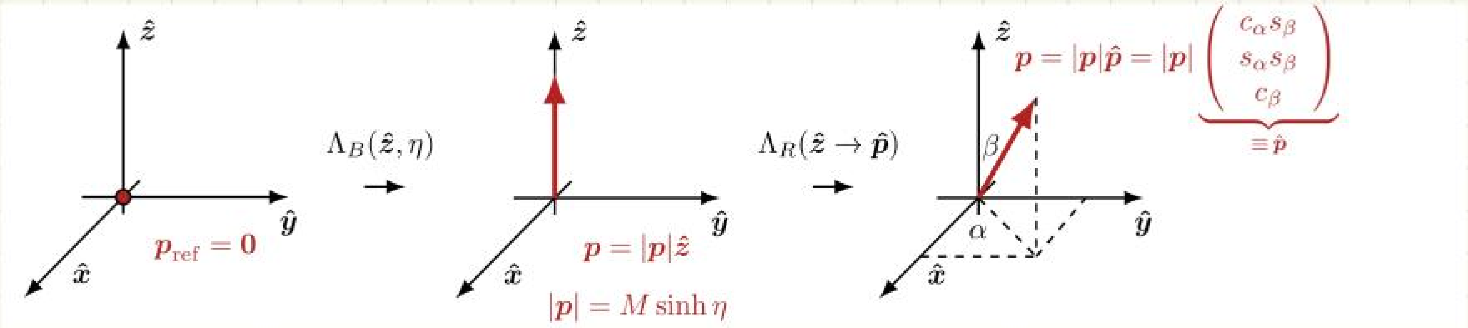
$$S_z^\mu = (0,0,0,1) \qquad \text{so that} \qquad S_z^\mu W_\mu = W_3 = -W^3$$

$$\longrightarrow \quad \boxed{-\frac{S_z^\mu W_\mu}{M} |M,\vec{0},s,\sigma\rangle = \sigma\, |M,\vec{0},s,\sigma\rangle}$$

The eigenvalue system so is:

$$\boxed{\begin{aligned} &P^\mu |M,\vec{0},s,\sigma\rangle = p^\mu_{ref} |M,\vec{0},s,\sigma\rangle \\ &P^2 |M,\vec{0},s,\sigma\rangle = M^2 |M,\vec{0},s,\sigma\rangle \\ &W^2 |M,\vec{0},s,\sigma\rangle = -Ms(s+1) |M,\vec{0},s,\sigma\rangle \\ &-\frac{S_z^\mu W_\mu}{M} |M,\vec{0},s,\sigma\rangle = \sigma\, |M,\vec{0},s,\sigma\rangle \end{aligned}} \qquad \begin{aligned} &\text{with } s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots \\ &\text{and } -s \leq \sigma \leq s \quad \text{for each } s \end{aligned}$$

STEP 5 (Massive case) Constructing the states with generic momentum



1) Start from P_{ref} and boost along $\hat{z}$ with rapidity η: $\Lambda_B(\hat{z},\eta) = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}$

$$\Lambda_B(\hat{z},\eta)\, P_{ref} = \Lambda_B(\hat{z},\eta)\begin{pmatrix} M \\ \vec{0} \end{pmatrix} = \begin{pmatrix} M\cosh\eta \\ 0 \\ 0 \\ M\sinh\eta \end{pmatrix} \qquad |\vec{p}| = M|\sinh\eta|$$

2) Apply a rotation from $\hat{z}$ to a certain $\hat{p} = \begin{pmatrix} c\alpha\, s\beta \\ s\alpha\, s\beta \\ c\beta \end{pmatrix}$

$$\Lambda_R(\hat{z}\to\hat{p})\cdot\Lambda_B(\hat{z},\eta)\, P_{ref} = \begin{pmatrix} M\cosh\eta \\ |\vec{p}|\,\hat{p} \end{pmatrix}$$

This combination of transformations depends on 3 parameters η and, α and β that define $\hat{p}$. We can substitute η with $\frac{|p|}{M}$ and define:

$$\boxed{H\left(\hat{p},\frac{|\vec{p}|}{M}\right) =: \Lambda_R(\hat{z}\to\hat{p})\,\Lambda_B(\hat{z},\eta)} \qquad \text{such that} \quad H\left(\hat{p},|\vec{p}|/M\right)^{\mu}{}_{\nu}\, P^{\nu}_{ref} = p^{\mu}$$

with a generic $p^{\mu} = \left(\sqrt{|\vec{p}|^2+M^2}, |\vec{p}|\,\hat{p}\right)$. N.B. H is a 4x4 matrix that acts on 4 vector.

We now introduce the operator U_H which corresponds to the transformation H in the representatio of L.G. that acts on the state $|M,\vec{0},s,\sigma\rangle$

How U_H operates?

Let's consider $U_H|M,\vec{0},s,\sigma\rangle$. Since P^2 and W^2 are Casimir.

$$P^2\, U_H|M,\vec{0},s,\sigma\rangle = M^2\, U_H|M,\vec{0},s,\sigma\rangle$$

$$W^2\, U_H|M,\vec{0},s,\sigma\rangle = -M^2 s(s+1)\, U_H|M,\vec{0},s,\sigma\rangle$$

What about P^{μ}?

$$P^{\mu}\,U_H|M,\vec{0},s,\sigma\rangle = U_H U_H^{-1} P^{\mu} U_H|M,\vec{0},s,\sigma\rangle =$$

$$= U_H\, H^{\mu}{}_{\nu}\, P^{\nu}|M,\vec{0},s,\sigma\rangle =$$

$$= U_H\, H^{\mu}{}_{\nu}\, P^{\nu}_{ref}|M,\vec{0},s,\sigma\rangle =$$

$$= U_H\, p^{\mu}|M,\vec{0},s,\sigma\rangle =$$

$$= p^{\mu}\, U_H|M,\vec{0},s,\sigma\rangle$$

Therefore $U_H|M,\vec{0},s,\sigma\rangle$ is an eigenstate of P^{μ} with eigenvalue p^{μ} (obtained by $H^{\mu}{}_{\nu}P^{\nu}_{ref}$.

Consequently we define the state:

$$\boxed{|M,\vec{p},s,\sigma\rangle =: U_H\left(\hat{p},\frac{|\vec{p}|}{M}\right)|M,\vec{0},s,\sigma\rangle} \; \bigstar$$

Therefore we can rewrite the system of eigenvalue equations as:

$$\boxed{\begin{array}{l} P^\mu \,|M,\vec{p},s,\sigma\rangle = p^\mu \,|M,\vec{p},s,\sigma\rangle \\ P^2 \,|M,\vec{p},s,\sigma\rangle = M^2 \,|M,\vec{p},s,\sigma\rangle \\ W^2 \,|M,\vec{p},s,\sigma\rangle = -Ms(s+1)\,|M,\vec{p},s,\sigma\rangle \\ \\ \text{Wait, what about the } 4^{th} \text{ equation?} \end{array}}$$

Let me define:

$$S_p^\mu =: H^\mu{}_\nu S_z^\nu \quad ; \qquad S_z^\mu = (H^{-1})^\mu{}_\nu S_p^\nu = H_\nu{}^\mu S_p^\nu$$

therefore:

$$-\frac{S_p^\mu W_\mu}{M}\,|M,\vec{p},s,\sigma\rangle = -\frac{S_p^\mu}{M} W_\mu\, \mathcal{U}_H \,|M,\vec{0},s,\sigma\rangle =$$

$$= -\frac{S_p^\mu}{M}\, \mathcal{U}_H \mathcal{U}_H^{-1} W_\mu \mathcal{U}_H \,|M,\vec{0},s,\sigma\rangle =$$

$$= -\frac{S_p^\mu}{M}\, \mathcal{U}_H\, H_\mu{}^\nu W_\nu \,|M,\vec{0},s,\sigma\rangle =$$

$$= \mathcal{U}_H \left(\frac{-S_p^\mu}{M}\right)(H^{-1})^\nu{}_\mu W_\nu \,|M,\vec{0},s,\sigma\rangle =$$

$$= \mathcal{U}_H\, \sigma \,|M,\vec{0},s,\sigma\rangle$$

$$\longrightarrow \quad -\frac{S_p^\mu W_\mu}{M}\,|M,\vec{p},s,\sigma\rangle = \sigma\,|M,\vec{p},s,\sigma\rangle$$

Therefore we can add it to the system:

$$\boxed{\begin{array}{l} P^\mu \,|M,\vec{p},s,\sigma\rangle = p^\mu \,|M,\vec{p},s,\sigma\rangle \\ P^2 \,|M,\vec{p},s,\sigma\rangle = M^2 \,|M,\vec{p},s,\sigma\rangle \\ W^2 \,|M,\vec{p},s,\sigma\rangle = -Ms(s+1)\,|M,\vec{p},s,\sigma\rangle \\ -\frac{S_p^\mu W_\mu}{M}\,|M,\vec{p},s,\sigma\rangle = \sigma\,|M,\vec{p},s,\sigma\rangle \end{array}}$$

Compute the action of $\frac{-S_p^\mu W_\mu}{M}$ on $|M,\vec{p},s,\sigma\rangle$ explicitly:

<u>1st step</u>: Compute S_p^μ

$$S_p^\mu =: H^\mu{}_\nu S_z^\nu \longrightarrow S_p = \Lambda_R(\hat{z}\to\hat{p})\,\Lambda_B(\hat{z},\eta)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \Lambda_R(\hat{z}\to\hat{p})\begin{pmatrix}\sinh\eta\\0\\0\\\cosh\eta\end{pmatrix} = \begin{pmatrix}\sinh\eta\\ \hat{p}\cosh\eta\end{pmatrix}$$

Since $\sinh\eta = \frac{|\vec{p}|}{M}$, $\cosh^2\eta = 1+\sinh^2\eta = 1+\frac{|\vec{p}|^2}{M^2} = \frac{p^{0\,2}}{M^2}$

$$\longrightarrow S_p^\mu = \begin{pmatrix}\frac{|\vec{p}|}{M}\\[4pt] \frac{p^0\hat{p}}{M}\end{pmatrix} = \begin{pmatrix}\frac{|\vec{p}|}{M}\\[4pt] \frac{p^0\vec{p}}{|\vec{p}|M}\end{pmatrix}$$

<u>2nd step</u>: Compute $-\frac{S_p^\mu W_\mu}{M}$: (detail on 🔗)

$$\longrightarrow \frac{-S_p^\mu W_\mu}{M} = -\frac{1}{M^2}\left[|\vec{p}|\vec{J}\cdot\vec{p} - \frac{p^0\vec{p}}{|\vec{p}|}\cdot\left(\vec{J}\cdot p^0 + \vec{K}\times\vec{P}\right)\right]$$

We're interested to the action of $-\frac{S_p^\mu W_\mu}{M}|M,\vec{p},s,\sigma\rangle$:

$$\longrightarrow -\frac{S_p^\mu W_\mu}{M}|M,\vec{p},s,\sigma\rangle = -\frac{1}{M}\left[|\vec{p}|\vec{J}\cdot\vec{p} - \frac{(p^0)^2\,\vec{p}\cdot\vec{J}}{|\vec{p}|}\right]|M,\vec{p},s,\sigma\rangle =$$

$$= -\frac{1}{M^2}\frac{1}{|\vec{p}|}\Big[\underbrace{|\vec{p}|^2-(p^0)^2}_{=M^2}\Big](\vec{p}\cdot\vec{J})\,|M,\vec{p},s,\sigma\rangle =$$

$$= \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}|M,\vec{p},s,\sigma\rangle$$

$$\longrightarrow \boxed{-\frac{S_p^\mu W_\mu}{M}|M,\vec{p},s,\sigma\rangle = \frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}|M,\vec{p},s,\sigma\rangle}$$

Therefore we understand that σ represents the projection of the spin along the direction of motion. This is the "so called" helicity.

Notice that strictly speaking helicity is the projection of the tot. ang. momentum J along the dir. of motion. However along the dir. of $\vec{p}$ the orbital part vanishes and we are only sensitive to the spin part.

Change of notation : $\boxed{\sigma\leftrightarrow\lambda}$ (universal notation for the helicity).

$$\longrightarrow \boxed{\begin{aligned} &P^\mu|M,\vec{p},s,\lambda\rangle = p^\mu|M,\vec{p},s,\lambda\rangle \\ &P^2|M,\vec{p},s,\lambda\rangle = M^2|M,\vec{p},s,\lambda\rangle \\ &W^2|M,\vec{p},s,\lambda\rangle = -M s(s+1)|M,\vec{p},s,\lambda\rangle \\ &\frac{\vec{J}\cdot\vec{p}}{|\vec{p}|}|M,\vec{p},s,\lambda\rangle = \lambda|M,\vec{p},s,\lambda\rangle \end{aligned}}$$

STEP 6 (Massive case) Definition of unitary irreps. of the Poincare' group

Theorem (Unitary irreps of Poincare' group)

The subspace generated by $\{|M,\vec{p},s,\lambda\rangle\}_{(M,s\text{ fixed})}$ gives a unitary irreducible representation of Poincare' group. (The representation is ∞ dimensional as expected for unitary repr. of a non-compact Lie Group).

Proof:

Consider a Poincare' transformation (combination of L.T. & space-time translation).

$$\mathcal{P}(\Lambda,a) = \mathcal{P}(\mathbb{1},a)\cdot\mathcal{P}(\Lambda,0)$$

We indicate with:

$$\mathcal{U}(\Lambda,a) = \mathcal{U}(\mathbb{1},a)\cdot\mathcal{U}(\Lambda,0)$$

the corresponding operator acting on $|M,\vec{p},s,\lambda\rangle$

1st step How $\mathcal{U}(\mathbb{1},a)$ acts on $|M,\vec{p},s,\lambda\rangle$

We Know that: $\mathcal{U}(\mathbb{1},a) = e^{i a_\mu P^\mu}$

$$\longrightarrow \boxed{\mathcal{U}(\mathbb{1},a)\,|M,\vec{p},s,\lambda\rangle = e^{-i a_\mu P^\mu}|M,\vec{p},s,\lambda\rangle = e^{-i a\cdot p}|M,\vec{p},s,\lambda\rangle}$$

2nd step How $\mathcal{U}(\Lambda,0)$ acts on $|M,\vec{p},s,\lambda\rangle$

Let's consider the state $\mathcal{U}(\Lambda,0)|M,\vec{p},s,\lambda\rangle$ and let's act on it with P^μ:

$$\longrightarrow P^\mu\,\mathcal{U}(\Lambda,0)|M,\vec{p},s,\lambda\rangle = \mathcal{U}(\Lambda,0)\,\mathcal{U}(\Lambda,0)^{-1}P^\mu\,\mathcal{U}(\Lambda,0)|M,\vec{p},s,\lambda\rangle$$

$$= \mathcal{U}(\Lambda,0)\,\Lambda^\mu{}_\nu P^\nu|M,\vec{p},s,\lambda\rangle$$

$$= \mathcal{U}(\Lambda,0)\,\Lambda^\mu{}_\nu p^\nu|M,\vec{p},s,\lambda\rangle$$

We now define:

$$p^\mu_\Lambda =: \Lambda^\mu{}_\nu p^\nu$$

$$\longrightarrow P^\mu\,\mathcal{U}(\Lambda,0)|M,\vec{p},s,\lambda\rangle = p^\mu_\Lambda\,\mathcal{U}(\Lambda,0)|M,\vec{p},s,\lambda\rangle$$

Therefore $\mathcal{U}(\Lambda,0)|M,\vec{p},s,\lambda\rangle$ is an eigenstate of P^μ with eigenvalue p^μ_Λ.

Based of ★🔗., the eigenstate of P^μ with eigenvalue p^μ_Λ can be also written as:

$$\longrightarrow \boxed{|M,\vec{p}_\Lambda,s,\lambda\rangle =: \mathcal{U}_H\left(\hat{p}_\Lambda,\frac{|\vec{p}_\Lambda|}{M}\right)|M,\vec{0},s,\lambda\rangle}$$

The Key point is to understand the relation between ● and ●

$$\mathcal{U}(\Lambda,0)|M,\vec{p},s,\lambda\rangle = \mathcal{U}(\Lambda,0)\,\mathcal{U}_H\left(\hat{p},\frac{|\vec{p}|}{M}\right)|M,\vec{0},s,\lambda\rangle =$$

$$= \mathcal{U}_H\left(\hat{p}_\Lambda,\frac{|\vec{p}_\Lambda|}{M}\right)\mathcal{U}_H^{-1}\left(\hat{p}_\Lambda,\frac{|\vec{p}_\Lambda|}{M}\right)\mathcal{U}(\Lambda,0)\,\mathcal{U}_H\left(\hat{p},\frac{|\vec{p}|}{M}\right)|M,\vec{0},s,\lambda\rangle \quad ★$$

Let's focus on

In the space of 4-momenta the yellow operator on $|M,\vec{0},s,\lambda\rangle$ reads:

$$\begin{tikzcd} p^\mu_{ref} \arrow[r, "{H(\hat{p},\,|\vec{p}|/M)}"] \arrow[rrr, bend right] & p^\mu = H^\mu{}_\nu p^\nu_{ref} \arrow[r, "\Lambda"] & p^\mu_\Lambda = \Lambda^\mu{}_\nu p^\nu \arrow[r, "{H(\hat{p}_\Lambda,\frac{|\vec{p}_\Lambda|}{M})^{-1}}"] & H\left(\hat{p}_\Lambda,\frac{|\vec{p}_\Lambda|}{M}\right)^{-1}p_\Lambda = p_{ref} \end{tikzcd}$$

p_{ref} is invariant ⟶ the chain of transformations $\in G(p)$

Wigner rotation :
$$\boxed{\omega(\Lambda,p) \overset{def}{=} H\left(\hat{p}_\Lambda,\frac{|\vec{p}_\Lambda|}{M}\right)^{-1}\Lambda\, H\left(\hat{p},\frac{|\vec{p}|}{M}\right)} \in \text{Little group.}$$

$\omega(\Lambda,p)$ is a 4-dim matrix acting on 4-vector, consequently

$$D_s(\omega(\Lambda,p)) = \mathcal{U}_H\left(\hat{p}_\Lambda,\frac{|\vec{p}_\Lambda|}{M}\right)^{-1}\mathcal{U}(\Lambda,0)\,\mathcal{U}_H\left(\hat{p},\frac{|\vec{p}|}{M}\right)$$

represents the action of a Wigner rotation on $|M,\vec{0},s,\lambda\rangle$

$$\longrightarrow \bigstar = \mathcal{U}_H\left(\hat{p}_\Lambda, \frac{|\vec{p}_\Lambda|}{M}\right) \sum_{\lambda'} \left[D_s(\omega(\Lambda,p))\right]_{\lambda'\lambda} |M, \vec{0}, s, \lambda'\rangle =$$

$$= \sum_{\lambda'} \left[D_s(\omega(\Lambda,p))\right]_{\lambda'\lambda} |M, \vec{p}_\Lambda, s, \lambda'\rangle =$$

$$\longrightarrow \boxed{\mathcal{U}(\Lambda,0)\,|M,\vec{p},s,\lambda\rangle = \sum_{\lambda'} \left[D_s(\omega(\Lambda,p))\right]_{\lambda'\lambda} |M, \vec{p}_\Lambda, s, \lambda'\rangle}$$

We now combine the 2 results as $\mathcal{U}(\Lambda,a) = \mathcal{U}(\mathbb{1},a)\cdot\mathcal{U}(\Lambda,0)$

$$\longrightarrow \boxed{\mathcal{U}(\Lambda,a)\,|M,\vec{p},s,\lambda\rangle = e^{i a_\mu P^\mu_\Lambda} \sum_{\lambda'} \left[D_s(\omega(\Lambda,p))\right]_{\lambda'\lambda} |M, \vec{p}_\Lambda, s, \lambda'\rangle} \quad \text{Key equation}$$

Comments:

* The Key eq. specifies the action of a generic Poincaré transformation on $|M,\vec{p},s,\lambda\rangle$ in the subspace $\{|M,\vec{p},s,\lambda\rangle\}_{(M,S.\ \text{fixed})}$

* The map $\mathcal{P}(\Lambda,a) \longrightarrow \mathcal{U}(\Lambda,a)$ is an homomorphism.

$$P(\Lambda_2,a_2)\cdot P(\Lambda_1,a_1) = P(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \longrightarrow \mathcal{U}(\Lambda_2,a_2)\,\mathcal{U}(\Lambda_1,a_1) = \mathcal{U}(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

* Therefore if we limit to the subspace $\{|M,\vec{p},s,\lambda\rangle\}_{(M,S\ \text{fixed})}$ the representation is irreducible i.e. it's impossible to find a subspace invariant under Poincaré transformation.

* Finally the representation is also unitary. To check that we need to define an inner product

- $\langle M,\vec{p},s,\lambda|M,\vec{q},s,\lambda'\rangle \overset{\text{def}}{=} (2\pi)^3\delta_{\lambda\lambda'}(2E_p)\delta(\vec{p}-\vec{q})$
- $\mathbb{1} = \sum_{\lambda=1}^{2S+1}\int \frac{d^3\vec{p}}{(2\pi)^3(2E_p}|M,\vec{p},s,\lambda\rangle\langle M,\vec{p},s,\lambda|$

$$\mathcal{U}(\Lambda,a)\,\mathcal{U}(\Lambda,a)^\dagger = \mathcal{U}(\Lambda,a)\sum_\lambda \int\frac{d^3p}{(2\pi)^3 2E_p}|M,\vec{p},s,\lambda\rangle\langle M,\vec{p},s,\lambda|\,\mathcal{U}(\Lambda,a)^\dagger =$$

$$= \sum_\lambda^{2S+1}\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}\left[\mathcal{U}(\Lambda,a)|M,\vec{p},s,\lambda\rangle\right]\left[\langle M,\vec{p},s,\lambda|\,\mathcal{U}(\Lambda,a)^\dagger\right] =$$

$$= \sum_{\lambda,\eta,\tilde{\eta}}\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}\left[D_s(\omega(\Lambda,p))\right]_{\eta\lambda}\left[D_s(\omega(\Lambda,p))\right]^*_{\lambda\tilde{\eta}}|M,\vec{p}_\Lambda,s,\eta\rangle\langle M,\vec{p}_\Lambda,s,\tilde{\eta}| =$$

$$= \sum_{\eta\tilde{\eta}}\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}\delta_{\eta\tilde{\eta}}\,|M,\vec{p}_\Lambda,s,\eta\rangle\langle M,\vec{p}_\Lambda,s,\tilde{\eta}| =$$

$$= \sum_\eta\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p}|M,\vec{p}_\Lambda,s,\eta\rangle\langle M,\vec{p}_\Lambda,s,\eta| \longrightarrow \mathcal{U}(\Lambda,a)\,\mathcal{U}(\Lambda,a)^\dagger = \mathbb{1}$$

* the representation is ∞ dimensional as expected $\vec{p}\in\mathbb{R}^3$. □

Conclusion:

- The states $\{|M,\vec{p},s,\lambda\rangle\}_{(M,S)}$ represent a Q.M. free particle with mass M and spin s. Such particle has $2s+1$ D.o.f. which are described by the helicity quantum number. The action of $\mathcal{P}$. transf on these states occurs via unitary operators → therefore in according with Wigners theorem $\mathcal{P}$. transf represent a symmetry of the free particle system.

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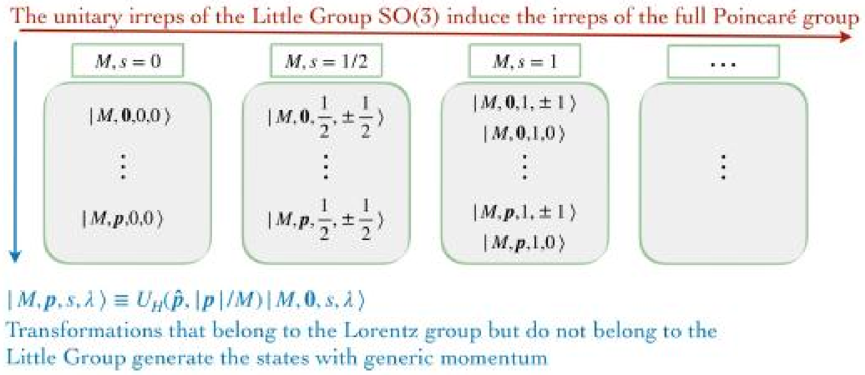


Figura 1.6: *Unitary (infinite-dimensional) irreducible representations of the Poincaré group in the massive case.*

- The helicity is a good quantum number since commutes with P^0 (energy, i.e. Hamiltonian of the free particle). Properties:

1) For a massive particle the helicity is not Lorentz invariant.

2) It is invariant under pure rotations. Intuitive proof: $h \propto \vec{J}\cdot\vec{P}$ → therefore a rotation by def. leaves the scalar product untouched.

Proof:

Consider a pure spatial rotation Λ_R. Let's compute $\mathcal{U}(\Lambda_R)|M,\vec{P},s,\lambda\rangle$:

$$\mathcal{U}(\Lambda_R)|M,\vec{P},s,\lambda\rangle = \sum_{\lambda'}\left[D_s(\omega(\Lambda_R,P))\right]_{\lambda'\lambda}|M,\vec{P}_{\Lambda_R},s,\lambda'\rangle \qquad \text{where} \quad P^{\mu}_{\Lambda_R} = \Lambda_R{}^{\mu}{}_{\nu}P^{\nu} = (P^0_{\Lambda}, \vec{P}_{\Lambda_R}) \quad \Lambda_R = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right)$$

$$\omega(\Lambda_R,P) \overset{\text{def}}{=} H\left(\hat{P}_{\Lambda_R}, \frac{|\vec{P}_{\Lambda_R}|}{M}\right)^{-1}\Lambda_R\, H\left(\hat{P}_{\Lambda_R}, \frac{|\vec{P}_{\Lambda_R}|}{M}\right) \quad \text{with} \quad H\left(\hat{P}, \frac{|\vec{P}|}{M}\right) \overset{\text{def}}{=} \Lambda_R(\hat{z}\to\hat{P})\Lambda_B\left(\hat{z},\frac{|\vec{P}|}{M}\right)$$

Since Λ_R is a pure rotation $|P_{\Lambda_R}| = |P|$. Consequently:

$$\omega(\Lambda_R,P) = \Lambda_B\left(\hat{z},\frac{|\vec{P}|}{M}\right)^{-1}\Lambda_R(\hat{z}\to\hat{P}_{\Lambda_R})^{-1}\Lambda_R\,\Lambda_R(\hat{z}\to\hat{P})\,\Lambda_B\left(\hat{z},\frac{|\vec{P}|}{M}\right)$$

Chain of rotations ⟶ rotation
It rotates $\hat{z} \longrightarrow \hat{z}$ (leaves the vector along $\hat{z}$ invariant)
⟶ it has to be a rotation around $\hat{z}$ → $\equiv e^{-i\alpha J^3_{4vec}}$

$$\Lambda_R(\hat{z}\longrightarrow\hat{P}_{\Lambda_R})^{-1}\Lambda_R\,\Lambda_R(\hat{z}\longrightarrow\hat{P}) \equiv e^{-i\alpha J^3_{4vec}} \quad \text{for some angle } \alpha = \alpha(\Lambda_R,P)$$

$$\longrightarrow \left[D_s(\hat{z},\alpha)\right]_{\lambda'\lambda} = \langle s,\lambda'|e^{-i\alpha J^3_{4vec}}|s,\lambda\rangle =$$

but J^3 is diagonal → the exp of a diag matrix is diag

$$\longrightarrow \quad = e^{-i\alpha\lambda}\delta_{\lambda\lambda'}$$

Therefore $\delta_{\lambda\lambda'} \Rightarrow \lambda = \lambda'$ (helicity invariant under rotation) □

Comment:

As we know step s can be implemented in different ways. We constructed p by boosting and rotating. We could try to follow another way:

We directly apply a boost along $\hat{p}$ direction: $\boxed{\Lambda_B(\hat{p},\eta)\, P_{ref} = \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix}}$

where η and p are related through $\cosh\eta = \frac{p^0}{M}$; $\sinh\eta = \frac{|\vec{p}|}{M}$

We can use the Weimberg notation $\boxed{\Lambda_B(\hat{p},\eta) =: L(\hat{p}, |\vec{p}|/M)}$ 4×4 matrix that acts on 4-vectors

The procedure to construct the representation of P.G. is similar (athough there are a couple of difference)

Let's introduce

$$\mathcal{U}_L \equiv \mathcal{U}_L(\hat{p}, |\vec{p}|/M)$$

it is the operator that corresponds to the transformation L in the representation of L.G. that acts on states $|M, \vec{0}, s, \sigma\rangle$

We define the state $|M, \vec{p}, s, \sigma\rangle$:

$$|M, \vec{p}, s, \sigma\rangle \overset{def}{=} \mathcal{U}_L(\hat{p}, |\vec{p}|/M)\,|M, \vec{0}, s, \sigma\rangle$$

it is still eigenstate of P^μ, P^2, W^2. σ is not the helicity; to understand its physical meaning we define:

$$S_L^\mu =: L^\mu{}_\nu S_z^\nu \quad ; \quad S_z^\mu = (L^{-1})^\mu{}_\nu S^\nu{}_L = L_\nu{}^\mu S^\nu_L$$

$$\longrightarrow \quad -\frac{S_L^\mu W_\mu}{M}|M, \vec{p}, s, \sigma\rangle = \sigma|M, \vec{p}, s, \sigma\rangle$$

Therefore σ is the eigenvalue of $-\frac{S_L^\mu W^\mu}{M} \neq$ helicity operator. What is the meaning of this operator? We can write it more explicitly:

$$-\frac{S_L^\mu W_\mu}{M} = \frac{-L^\mu{}_\nu S_z^\nu W_\mu}{M} = -\frac{1}{M} L^\mu{}_3 W_\mu = \frac{1}{M} L_\mu{}^3 W^\mu = \frac{1}{M}(L^{-1})^3{}_\mu W^\mu$$

We can define:

$$\boxed{J^K \overset{def}{=} \frac{1}{M}[L^{-1}]^K{}_\mu W^\mu}$$

(we can check that $\underbrace{[J^i, J^j] = i\,\varepsilon_{ijk} J^K}_{\text{algebra of ang. mom}}$ and $\underbrace{[J^i, P^K] = 0}_{\text{property of spin}}$ $\longrightarrow$ J^K describes the spin operator

$$\longrightarrow \quad \boxed{J^3\,|M, \vec{p}, s, \sigma\rangle = \sigma\,|M, \vec{p}, s, \sigma\rangle}$$

In conclusion the eigenvalue system is now given by:

$$\boxed{\begin{aligned} P^\mu\,|M,\vec{p},s,\sigma\rangle &= p^\mu\,|M,\vec{p},s,\sigma\rangle \\ P^2\,|M,\vec{p},s,\sigma\rangle &= M^2\,|M,\vec{p},s,\sigma\rangle \\ W^2\,|M,\vec{p},s,\sigma\rangle &= -M^2 s(s+1)\,|M,\vec{p},s,\sigma\rangle \\ J^3\,|M,\vec{p},s,\sigma\rangle &= \sigma\,|M,\vec{p},s,\sigma\rangle \end{aligned}} \qquad \text{with } s=0,\tfrac{1}{2},1,\dots \quad -s\le\sigma\le s$$

Conventionally we call:

$$\boxed{\begin{aligned} &|M,\vec{p},s,\lambda\rangle : \ \text{helicity basis} \rightarrow \lambda:\ \text{helicity} \\ &|M,\vec{p},s,\sigma\rangle : \ \text{spin basis} \rightarrow \sigma:\ \text{projection of spin along } \hat{z} \end{aligned}}$$

There is a little difference in the Wigner rotation. If we take $\mathcal{U}(\Lambda,0)\,|M,\vec{p},s,\sigma\rangle$; remains valid that:

$$P^\mu\,\mathcal{U}(\Lambda,0)\,|M,\vec{p},s,\sigma\rangle = p^\mu_\Lambda\,\mathcal{U}(\Lambda)\,|M,\vec{p},s,\sigma\rangle \longrightarrow |M,\vec{p},s,\sigma\rangle = \mathcal{U}_L\!\left(\hat{p}_\Lambda,\tfrac{|\vec{p}_\Lambda|}{M}\right)|M,\vec{0},s,\sigma\rangle$$

$$\longrightarrow \mathcal{U}(\Lambda)\,|M,\vec{p},s,\sigma\rangle = \mathcal{U}(\Lambda)\,\mathcal{U}_L\!\left(\hat{p},\tfrac{|\vec{p}|}{M}\right)|M,\vec{0},s,\sigma\rangle =$$

$$= \mathcal{U}_L\!\left(\hat{p}_\Lambda,\tfrac{|\vec{p}_\Lambda|}{M}\right)\,\mathcal{U}_L\!\left(\hat{p}_\Lambda,\tfrac{|\vec{p}_\Lambda|}{M}\right)^{-1}\mathcal{U}(\Lambda)\,\mathcal{U}_L\!\left(\hat{p},\tfrac{|\vec{p}|}{M}\right)|M,\vec{0},s,\sigma\rangle$$

We now focus on

In the space of 4 momenta :

$$\begin{tikzcd} p^\mu_{ref} \arrow[r, "L(\hat{p}{,}\frac{|\vec{p}|}{M})"] \arrow[rrr, bend right] & p^\mu = L^\mu{}_\nu\, p^\nu_{ref} \arrow[r, "\Lambda"] & p^\mu_\Lambda = \Lambda^\mu{}_\nu\, p^\nu \arrow[r, "L(\hat{p}_\Lambda{,}\frac{|\vec{p}_\Lambda|}{M})^{-1}"] & p^\mu_{ref} \end{tikzcd}$$

p_{ref} is invariant $\longrightarrow$ the chain $\in G(p)$

$$\longrightarrow \boxed{\omega_L(\Lambda,p) \overset{def}{=} L\!\left(\hat{p}_\Lambda,\tfrac{|\vec{p}_\Lambda|}{M}\right)^{-1}\Lambda\, L\!\left(\hat{p},\tfrac{|\vec{p}|}{M}\right)}$$

therefore, after defining its version that acts on $|M,\vec{0},s,\sigma\rangle$, we can write that for a generic Poincaré transformation:

$$\boxed{\mathcal{U}(\Lambda,a)\,|M,\vec{p},s,\sigma\rangle = e^{i a\cdot p_\Lambda}\sum_{\sigma'}\left[D_s\left(\omega_L(\Lambda,p)\right)\right]_{\sigma'\sigma}|M,\vec{p}_\Lambda,s,\sigma'\rangle}$$

If Λ is a pure rotation $\Lambda=\Lambda_R$, therefore in this spin basis:

$$\boxed{\omega_L(\Lambda_R,p)=\Lambda_R}$$

MASSLESS CASE

In the massless case $p^2=0$; $p^0=|\vec{p}|>0$

STEP 3 (Massless case) Identification of a reference vector

We define the reference vector: $\boxed{p^\mu_{ref}\,(\kappa,0,0,\kappa) \quad \kappa>0}$ $\qquad \vec{p}_{ref} = \kappa\hat{z}$

Therefore the eigenvalue equations for P^μ and P^2 are

$$P^\mu|0,\kappa\hat{z}\rangle = p^\mu_{ref}\,|0,\kappa\hat{z}\rangle$$
$$P^2|0,\kappa\hat{z}\rangle = 0$$

STEP 4 (massless case) Identification of the Little Group

It seems that $G(p) = SO(2)$ (rotation around $\hat{z}$ axis). However it is almost correct indeed there are other symmetries that leave p_{ref} untouched. As we know, in fact, the Little Group has 3 generators, not just one. The 3 generators are given:

$$\vec{W} = \vec{J} + \frac{\vec{K}\times\vec{P}_{ref}}{P^0_{ref}} = \vec{J} + \frac{\vec{K}\times\cancel{\kappa}\hat{z}}{\cancel{\kappa}} = \vec{J}+\vec{K}\times\hat{z} = \vec{J}+K^2\hat{x}-K^1\hat{y} = (J^1+K^2,\, J^2-K^1,\, J^3)$$

In components we find the 3 generators

$$\boxed{T_x =: J^1+K^2 \;;\; T_y =: J^2-K^1 \;;\; T_z = J^3}$$

this is the one that generates rotation around $\hat{z}$-axis

We can now compute the algebra (commutation relations):

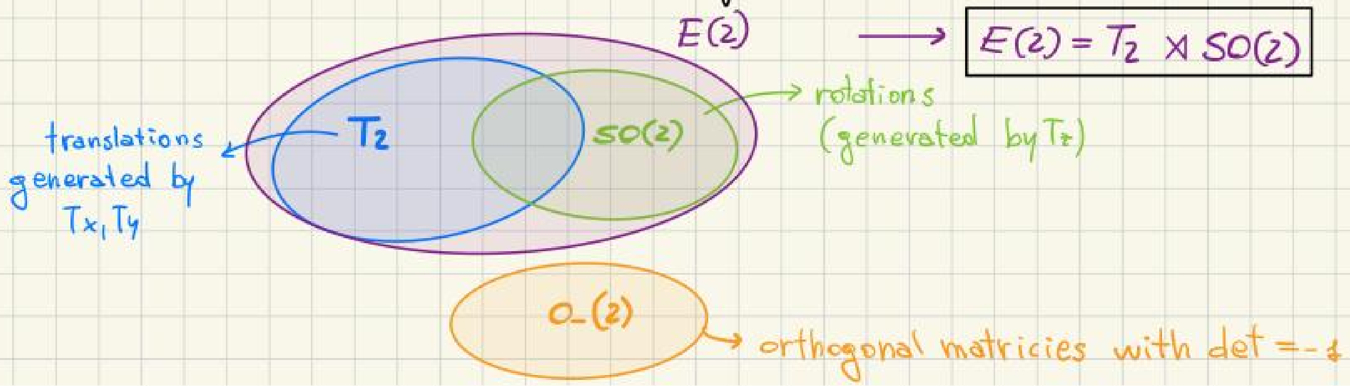
$$\boxed{\begin{aligned} &[T_x, T_y] = 0 \\ &[T_z, T_x] = i\,T_y \\ &[T_z, T_y] = -i\,T_x \end{aligned}}$$

Algebra of the Little Group

This algebra is the Lie Algebra of the 2-D Euclidean group $E(2)$

$$\boxed{G(P^\mu_{ref}) = E(2)}$$ Lille group in the massless case

The structure of $E(2)$ is the following:



$$E(2) \longrightarrow \boxed{E(2) = T_2 \rtimes SO(2)}$$

We can now proceed, as in the massive case, to induce a unitary representation of the Poincaré group starting from the irreducible representation of the little group. (E_2)

In the massive case we started from $|M,\vec{0}\rangle$ and found $|M,\vec{0},j,\sigma\rangle := |M,0\rangle \otimes |j,\sigma\rangle$ where $|j,\sigma\rangle$ are the common eigenstates of the Casimir $|J|^2$ and J^3.

In this massless case we start from $|M, K\hat{e}_3\rangle$. We need to find the analogue of the state $|j,\sigma\rangle$. To do that we see that $T_x^2 + T_y^2$ is a Casimir for E_2. Consequently:

$$\boxed{\begin{aligned} T_x|\vec{W}\rangle &= W_x|\vec{W}\rangle \\ T_y|\vec{W}\rangle &= W_y|\vec{W}\rangle \\ (T_x^2+T_y^2)|\vec{W}\rangle &= W^2|\vec{W}\rangle \end{aligned}}$$

$W^2 = W_x^2 + W_y^2 \geq 0$ (In the massive case P^2 can be negative)

We now need to add the rotational part generated by $J^3 = T_z$

J^3 is the generator of $SO(2)$. We constructed so far the irreps of $U(1) \cong SO(2)$. In terms of the Lie algebra of $U(1)$ each one of the irreps is identified by $\lambda \in \mathbb{Z}$

$$\longrightarrow J^3|\lambda\rangle = \lambda|\lambda\rangle$$

Since J^3 and $T_x^2 + T_y^2$ commute we diagonalize them together (in this way we are forced to neglet the equations for T_x and T_y because they do not commute with J^3).

$$(T_x^2+T_y^2)|W^2,\lambda\rangle = W^2|W^2,\lambda\rangle$$

$$J^3|W^2,\lambda\rangle = \lambda|W^2,\lambda\rangle$$

What are the possible values for λ and why we can write $J^3|\lambda\rangle = \lambda|\lambda\rangle$?

Let's forget about the Little group for a moment and consider $U(1) \cong SO(2)$

1) The group is abelian $\longrightarrow$ Shurs Lemma : all the complex irreps are 1D

2) The rules which define the group are simple:

$$\begin{cases} R(\theta_1)\cdot R(\theta_2) = R(\theta_1+\theta_2) \\ R(\theta=0) = e \\ R(\theta)^{-1} = R(-\theta) \end{cases}$$

$$R(\theta+2\pi) = R(\theta) \quad \text{(Compact group)}$$

3) The irreps are maps such that:

$$\begin{aligned} \mathcal{D} : U(1) &\longrightarrow GL(1,\mathbb{C}) \cong (\mathbb{C},\cdot) = \{z\in\mathbb{C} : z\neq 0\} \\ g &\longmapsto \mathcal{D}(g) \end{aligned}$$

Notation : $\mathcal{D}(R(\theta)) \equiv \mathcal{D}(\theta)$

$$\longrightarrow D'(\theta) = \lim_{\Delta\theta \to 0} \frac{D(\theta+\Delta\theta) - D(\theta)}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{D(\theta)\,D(\Delta\theta) - D(\theta)}{\Delta\theta} =$$

$$= D(\theta) \cdot \lim_{\Delta\theta \to 0} \frac{D(\Delta\theta) - D(0)}{\Delta\theta} = D(\theta) \cdot D'(0) \rightarrow = \text{const} \quad \tilde{c} \in \mathbb{C}$$

It's a differential equation : $D(\Delta\theta = 0) = 1 \longrightarrow D(\theta) = e^{\tilde{c}\cdot\theta}$

We need to impose $R(\theta + 2\pi) = R(\theta) \longrightarrow e^{\tilde{c}\theta} = e^{\tilde{c}\theta + 2\pi\tilde{c}} \longleftrightarrow \tilde{c} \in \mathbb{Z}$ ☐

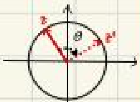
I call $\hat{c} \equiv \lambda$

Therefore, in conclusion:

$$\boxed{D(\theta) = e^{-i\lambda\theta} \quad \text{where } \lambda = 0, \pm 1, \pm 2, \pm 3, \dots}$$

- $\lambda = 0 \longrightarrow \mathcal{U}_0(\theta) = \mathbb{1}$ trivial representation

FAITHFUL REPRESENTATIONS (1 to 1 correspondene between elements of the group and operators)
- $\lambda = -1 \longrightarrow \mathcal{U}_{-1}(\theta) = e^{i\theta}$ rotates complex vectors (anticlockwise)
- $\lambda = +1 \longrightarrow \mathcal{U}_{+1}(\theta) = e^{-i\theta}$ rotates complex vectors (clockwise)



NOT FAITHFUL REPRESENTATIONS
- $\lambda = \pm m \rightarrow \mathcal{U}_{\pm m}(\theta) = e^{\mp i m\theta}$ $m \in \mathbb{Z} \setminus \{-1, 1\}$ rotate the $\mathbb{C}$ vectors m times

We can use the same construction from the point of view of the algebra. We Know that the group is abelian, the irreps are 1DIM and the group generator is J^3 (that we call for simplicity J). This irreps is also unitary and therefore J must be hermitian and then its eigenvalues are real.

$$\boxed{J|\lambda\rangle = \lambda|\lambda\rangle} \quad \lambda \in \mathbb{R} \qquad (\text{eigenvalue equation})$$

I can take the <u>exponential map</u>: $\longrightarrow \mathcal{U}(\theta) = e^{-i\theta J}$

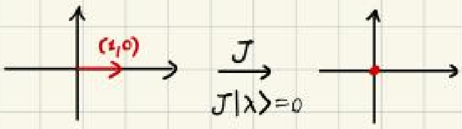
$$\longrightarrow \boxed{\mathcal{U}(\theta)|\lambda\rangle = e^{-i\theta J}|\lambda\rangle = e^{-i\theta\lambda}|\lambda\rangle}$$

$|\lambda\rangle$ is nothing but a basis in the representation space ($\mathbb{C}$). A good choice is:

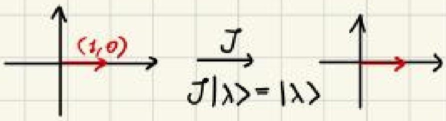
$$\boxed{|\lambda\rangle = (1,0)} \quad (\text{or } (0,1))$$

<u>What is the action of J on $|\lambda\rangle$</u>?

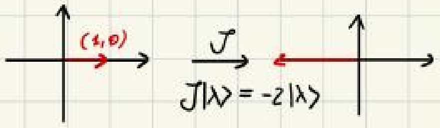
- $\lambda = 0$: trivial repr.



- $\lambda = 1$:



- $\lambda = -2$:



TRICKY PART

As we learnt in the massive case, we should consider projective representations of $U(1) \cong SO(2)$ allowed from a Q.M. setting.
In other words we should consider also $\lambda = \pm \frac{(2n+1)}{2}$ with $n \in \mathbb{N}$

$$\longrightarrow U_\lambda(\theta) = \exp\left(\pm i \frac{(2n+1)\theta}{2}\right)$$

If we combine two angles θ_1 and θ_2 such that $\theta_1 + \theta_2 = 2\pi$ we get:

$$U_\lambda(\theta_1) \cdot U_\lambda(\theta_2) = \exp\left(\pm i \frac{(2n+1)2\pi}{2}\right) = e^{i\pi} = -\mathbb{1}$$

therefore we have a projective representations. i.e. they do not give $\mathbb{1}$ when we rotate by 2π
In conclusion in addition to the integers value of λ we include also half integers.

$$\longrightarrow \boxed{\begin{aligned} &(T_2^2 + T_y^2)\,|W^2, \lambda\rangle = w^2 |W^2, \lambda\rangle \\ &J^3\,|W^2, \lambda\rangle = \lambda\,|W^2, \lambda\rangle \qquad \lambda = 0, \pm \tfrac{1}{2}, \pm 1, \pm \tfrac{3}{2}, \dots \end{aligned}}$$

A generic element of the Little Group $E(2)$ takes the form:

$$\boxed{D(\alpha, \beta, \theta) = e^{-i(\alpha T_x + \beta T_y + \theta J^3)}}$$

CRUCIAL POINT: we want to understand how the states $|w, \lambda\rangle$ form an irreducible representation of $E(2)$. We have to distinguish 2 possible cases:

- $\boxed{W^2 = 0}$ (Trivial case)

$$\rightarrow \begin{cases} T_x^2 + T_y^2\ |0, \lambda\rangle = 0 \\ T_x\ |0, \lambda\rangle = 0 \\ T_y\ |0, \lambda\rangle = 0 \end{cases} \longrightarrow D(\alpha, \beta, \theta) = e^{-i\alpha T_x - i\beta T_y - i\theta J^3} |0, \lambda\rangle = e^{-i\theta\lambda} |0, \lambda\rangle$$

The translational part trivializes to $\mathbb{1}$, while the action of rotations gives us only a phase. In other words the subspace $\{|0, \lambda\rangle\}_{\lambda \text{ fixed}}$ is left invariant without any chance of changing λ or creating a non-zero w.

$$\longrightarrow \boxed{\{|0, \lambda\rangle\}_{\lambda \text{ fixed}} \text{ unitary irreps } 1D \text{ of } E(2)}$$ (degenerate representations)
spatial transl $\rightarrow \mathbb{1}$

- $\boxed{W^2 > 0}$ (Tricky case)

$$\rightarrow \begin{cases} T_x\ |W^2, \lambda\rangle \neq 0 \\ T_y\ |W^2, \lambda\rangle \neq 0 \end{cases}$$

It's useful to define $T_\pm =: T_x \pm i T_y$. We can easily check that $[J^3, T_\pm] = \pm T_\pm$
Therefore:

$$J^3 T_\pm |W^2, \lambda\rangle = (T_\pm J^3 \pm T_\pm)|W^2, \lambda\rangle = (\lambda \pm 1) T_\pm |W^2, \lambda\rangle$$

The Key point is that T_x and T_y are lin. comb. of $T_\pm$ and therefore their action will rais or lower the value of λ $\longrightarrow$ if we apply an E_2 transformation we'll inevitably increase or decrease λ and force us to incorporate new values of λ. Consequently to obtain an irreps we are forced to consider <u>all</u> the possible value of λ

$\boxed{\left\{ |w^2, \lambda\rangle \right\}^{w^2 \text{ fixed}}_{\lambda = 0, \pm 1/2, \pm 1, \dots}}$: ∞ dim. unitary irreps of $E(2)$. (non degenerate repr.)

We can now finally write the system of eigenvalue equations:

$$\boxed{\begin{aligned}
&P^\mu\, |0, k\hat{z}, w^2, \lambda\rangle = p^\mu_{ref}\, |0, k\hat{z}, w^2, \lambda\rangle \\
&P^2\, |0, k\hat{z}, w^2, \lambda\rangle = 0 \\
&(T_x^2 + T_y^2)\, |0, k\hat{z}, w^2, \lambda\rangle = w^2\, |0, k\hat{z}, w^2, \lambda\rangle \\
&J^3\, |0, k\hat{z}, w^2, \lambda\rangle = \lambda\, |0, k\hat{z}, w^2, \lambda\rangle
\end{aligned}}$$

This is a well defined system of commuting operators:

1) - $[J^3, P^0] = 0$
   - $[J^3, P^j] = i\,\varepsilon_{ijk} P^k$, but if we restrict to the supspace formed by P_{ref} then the only non-vanishing comp. of P^j is P^3 $\longrightarrow [J^3, P^3] = 0$

2) If we restrict to the space formed by P_{ref} and consider the components of the Pauli Lubanski vector

$$W^0 = \vec{J}\cdot\vec{P} \qquad \vec{W} = \vec{J}P^0 + \vec{K}\times\vec{P}$$

$$\rightarrow \quad W^0\, |0, k\hat{z}, w^2, \lambda\rangle = \vec{J}\cdot\vec{P}\, |0, k\hat{z}, w^2, \lambda\rangle = k J^3\, |0, k\hat{z}, w^2, \lambda\rangle$$

$$\vec{W}\, |0, k\hat{z}, w^2, \lambda\rangle = (\vec{J}P^0 + \vec{K}\times\vec{P})\, |0, k\hat{z}, w^2, \lambda\rangle = k(\vec{J} + \vec{K}\times\hat{z})\, |0, k\hat{z}, w^2, \lambda\rangle = k\vec{W}\, |0, k\hat{z}, w^2, \lambda\rangle$$

$$\begin{aligned}
W^2\, |0, k, \hat{z}, w^2, \lambda\rangle &= k^2\left[(J^3)^2 - |\vec{W}|^2\right] |0, k\hat{z}, w^2, \lambda\rangle = k^2\left[(J^3)^2 - T_x^2 - T_y^2 - (J^3)^2\right] |0, k\hat{z}, w^2, \lambda\rangle = \\
&= -k^2(T_x^2 + T_y^2)\, |0, k\hat{z}, w^2, \lambda\rangle
\end{aligned}$$

Therefore we could rewrite the system as:

$$\boxed{\begin{aligned}
&P^\mu\, |0, k\hat{z}, w^2, \lambda\rangle = p^\mu_{ref}\, |0, k\hat{z}, w^2, \lambda\rangle \\
&P^2\, |0, k\hat{z}, w^2, \lambda\rangle = 0 \\
&-\frac{W^2}{k^2}\, |0, k\hat{z}, w^2, \lambda\rangle = w^2\, |0, k\hat{z}, w^2, \lambda\rangle \qquad w^2 \geq 0 \\
&\frac{W^3}{k}\, |0, k\hat{z}, w^2, \lambda\rangle = \lambda\, |0, k\hat{z}, w^2, \lambda\rangle \qquad \lambda = 0, \pm\tfrac{1}{2}, \pm 1, \dots
\end{aligned}}$$

Okay, now, if we now attempt to apply the method of induced representation to the massless case we have 2 alternatives:

i) We use degenerate irreps of E_2 $\{|0,\lambda\rangle\}_{\lambda \text{ fixed}}$ with $W^2=0$.

$$\longrightarrow \boxed{\{|0,\kappa\hat{z},0,\lambda\rangle\}_{\lambda \text{ fixed}} \text{ irreps of Poincaré group (massless case)}}$$

Phys. interpretation: eigenvalue of J^3 which is the projection of the tot. ang. mom. $\vec{J}$ along $\hat{z}$
→ it seems to be helicity

ii) We use non-degenerate irreps of E_2 $\{|0,\lambda\rangle\}^{W^2 \text{ fixed}}_{\lambda=0,\pm\frac{1}{2},\dots}$ $W^2>0$

$$\longrightarrow \boxed{\{|0,\kappa\hat{z},W^2,\lambda\rangle\}^{W^2 \text{ fixed}}_{\lambda=0,\pm\frac{1}{2},\pm1,\dots} \text{ irreps of Poincaré group (massless case)}}$$

Phy interpretation: less clear, 2 ambiguous points

1) for each repr. we are forced to use all the λ

2) different representations are continuously connected by the eigenvalue $W^2 \geq 0$: continuous spin representations.

We'll use $\{|0,\kappa\hat{z},W^2=0,\lambda\rangle\}_{\lambda \text{ fixed}}$

Notation:

$$\boxed{\begin{aligned} &P^\mu|0,\kappa\hat{z},\lambda\rangle = P^\mu_{ref}|0,\kappa\hat{z},\lambda\rangle \\ &P^2|0,\kappa\hat{z},\lambda\rangle = 0 \\ &-\frac{S_{\hat{z}}^\mu W_\mu}{\kappa}|0,\kappa\hat{z},\lambda\rangle = \lambda|0,\kappa\hat{z},\lambda\rangle \end{aligned}}$$

Remember also that $(T_x{}^2+T_y{}^2)|0,\kappa\hat{z},\lambda\rangle = T_x|0,\kappa\hat{z},\lambda\rangle = T_y|0,\kappa\hat{z},\lambda\rangle = 0$ (Important to understand Gauge invariance) ★

STEP 5 (massless case) Constructing the states with generic momentum

$$1)\quad \Lambda_B(\hat{z},\eta)\,P_{ref} = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}\begin{pmatrix} \kappa \\ 0 \\ 0 \\ \kappa \end{pmatrix} = \begin{pmatrix} \kappa(\cosh\eta+\sinh\eta) \\ 0 \\ 0 \\ \kappa(\cosh\eta+\sinh\eta) \end{pmatrix} = \begin{pmatrix} \kappa e^\eta \\ 0 \\ 0 \\ \kappa e^\eta \end{pmatrix} \equiv \begin{pmatrix} |P| \\ 0 \\ 0 \\ |P| \end{pmatrix}$$

$$2)\quad \Lambda_R(\hat{z}\to\hat{p})\,\Lambda_B(\hat{z},\eta)\,P_{ref} = \Lambda_R(\hat{z}\to\hat{p})\begin{pmatrix} |\vec{P}| \\ 0 \\ 0 \\ |\vec{P}| \end{pmatrix} = \begin{pmatrix} |\vec{P}| \\ \hat{P}|\vec{P}| \end{pmatrix}$$ still verifies massless mass shell condition $P^2=0$

I can define the transformation H (4 dim matrix acting on 4-vector)

$$H\left(\hat{p},\frac{|\vec{P}|}{\kappa}\right) =: \Lambda_R(\hat{z}\to\hat{p})\,\Lambda_B(\hat{z},\eta)$$

$$H\left(\hat{p},\frac{|\vec{P}|}{\kappa}\right)^\mu{}_\nu\,P^\nu_{ref} = P^\mu_{ref}$$

I define the operator that corresponds to H in the repr. of L.G. which acts on the states:

$$\mathcal{U}_H\left(\hat{p}, \frac{|\vec{p}|}{\kappa}\right) \equiv \mathcal{U}_H$$

- $\mathbb{P}^\mu\, \mathcal{U}_H|0,\kappa\hat{z},\lambda\rangle = p^\mu\, \mathcal{U}_H|0,\kappa\hat{z},\lambda\rangle$ → still eigenstate of $\mathbb{P}^\mu$

Therefore we could define:

$$\boxed{|0,\vec{p},\lambda\rangle \equiv \mathcal{U}_H\left(\hat{p}, \frac{|\vec{p}|}{\kappa}\right)|0,\kappa\hat{z},\lambda\rangle}$$

Meaning of λ in the moving frame

The idea is to introduce the operator:

$$S_p^\mu =: H^\mu{}_\nu S_z^\nu \quad ; \quad S_z^\mu = (H^{-1})^\mu{}_\nu S_p^\nu$$

Therefore we can write:

$$-\frac{S_p^\mu W_\mu}{\kappa}|0,\vec{p},\lambda\rangle = \lambda|0,\vec{p},\lambda\rangle \longrightarrow \lambda \text{ is the eigenvalue of } -\frac{S_p^\mu W_\mu}{\kappa}$$

What's the physical meaning of this operator?

We could write it more explicitly: $S_p^\mu = H^\mu{}_\nu S_z^\nu = (\sinh\eta, \hat{p}\cosh\eta)$

$$S_p^\mu W_\mu = S_p^0 W_0 + S_p^i W_i = S_p^0 W^0 - \vec{S}_p\cdot\vec{W} =$$

$$= \sinh\eta\,(\vec{J}\cdot\vec{P} - \hat{p}\cosh\eta\cdot(\vec{J}P^0 + \vec{K}\times\vec{P}) =$$

$$-\frac{S_p^\mu W_\mu}{\kappa}|0,\vec{P},\lambda\rangle = -\frac{1}{\kappa}\left[\sinh\eta\,\vec{J}\cdot\vec{P} - \hat{p}\cosh\eta\cdot(\vec{J}|\vec{P}| + \cancel{\vec{K}\times\vec{P}})\right] =$$

‼ when do the scalar p.

$$= -\frac{1}{\kappa}\left[\sinh\eta - \cosh\eta\right]\vec{J}\cdot\vec{P}|0,\vec{P},\lambda\rangle =$$

$$= +\frac{1}{\kappa}e^{-\eta}\,\vec{J}\cdot\vec{P}|0,\vec{P},\lambda\rangle$$

$$\checkmark \quad = \frac{1}{|\vec{P}|}\vec{J}\cdot\vec{P}|0,\vec{P},\lambda\rangle \longrightarrow \text{Helicity operator!}$$

Therefore we could write the system as:

$$\boxed{\begin{aligned} &\mathbb{P}^\mu|0,\kappa\hat{z},\lambda\rangle = P^\mu_{ref}|0,\kappa\hat{z},\lambda\rangle \\ &\mathbb{P}^2|0,\kappa\hat{z},\lambda\rangle = 0 \\ &\vec{J}\cdot\vec{P}|0,\kappa\hat{z},\lambda\rangle = \lambda|\vec{P}||0,\kappa\hat{z},\lambda\rangle \end{aligned}}$$

STEP 6 : (massless case) Definition of unitary irreps. of the Poincaré group

If we consider $\{|0, \vec{p}, \lambda\rangle\}_{\lambda \text{ fixed}}$ we need to understand the action of $\mathcal{U}(\Lambda, a)$ on these states

The strategy is to consider $\mathcal{U}(\Lambda, a) = \mathcal{U}(\mathbb{1}, a) \times \mathcal{U}(\Lambda, 0)$:

- $$\mathcal{U}(\mathbb{1}, a)|0, \vec{p}, \lambda\rangle = e^{i a_\mu P^\mu}|0, \vec{p}, \lambda\rangle = e^{i a \cdot p}|0, \vec{p}, \lambda\rangle$$
- $$P^\mu \mathcal{U}(\Lambda, 0)|0, \vec{p}, \lambda\rangle = p_\Lambda^\mu \, \mathcal{U}(\Lambda)|0, \vec{p}, \lambda\rangle \qquad p_\Lambda^\mu = \Lambda^\mu{}_\nu p^\nu \quad \text{(transformed eigenvalue)}$$

We could write : $|0, \vec{p}_\Lambda, \lambda\rangle \equiv \mathcal{U}_H(\hat{p}_\Lambda, \frac{|\vec{p}_\Lambda|}{k})|0, k\hat{z}, \lambda\rangle$

What is the relation between $|0, \vec{p}_\Lambda, \lambda\rangle$ and $\mathcal{U}(\Lambda)|0, \vec{p}, \lambda\rangle$?

1) $\mathcal{U}(\Lambda)|0, \vec{p}, \lambda\rangle = \mathcal{U}(\Lambda)\,\mathcal{U}_H(\hat{p}, \frac{|\vec{p}|}{k})|0, k\hat{z}, \lambda\rangle$

2) $\mathcal{U}_H(\hat{p}_\Lambda, \frac{|\vec{p}_\Lambda|}{k})\,\mathcal{U}_H(\hat{p}_\Lambda, \frac{|\vec{p}_\Lambda|}{k})^{-1}\mathcal{U}(\Lambda, 0)\,\mathcal{U}_H(\hat{p}, \frac{|\vec{p}|}{k})|0, k\hat{z}, \lambda\rangle$

Let's see what happen in the space of 4-momenta:

$$\begin{tikzcd} P_{ref} \arrow[r, "H(\hat{p}, \frac{|\vec{p}|}{k})"] \arrow[rrr, bend right] & p^\mu \arrow[r, "\Lambda"] & p_\Lambda^\mu \arrow[r, "H(\hat{p}_\Lambda, \frac{|\vec{p}_\Lambda|}{k})^{-1}"] & P_{ref} \end{tikzcd}$$

it leaves P_{ref} invariant
→ element of SE(2) (Little Group)

This means that :

because of ★

$$\mathcal{U}_H(\hat{p}_\Lambda, \frac{|\vec{p}_\Lambda|}{k})^{-1}\mathcal{U}(\Lambda, 0)\,\mathcal{U}_H(\hat{p}, \frac{|\vec{p}|}{k}) = e^{\cancel{-i\alpha(\Lambda, p)T_x} \cancel{-i\beta(\Lambda, p)T_y} - i\theta(\Lambda, p)J^3}$$

$$e^{-i\theta(\Lambda, p)\lambda}|0, k\hat{z}, \lambda\rangle =$$

$$= e^{-i\theta(\Lambda, p)\lambda}|0, \vec{p}_\Lambda, \lambda\rangle$$

$\boxed{\theta(\Lambda, p) \text{ Wigner phase}}$

In conclusion:

$$\boxed{\mathcal{U}(\Lambda, a)|0, \vec{p}, \lambda\rangle = e^{i a_\mu p_\Lambda^\mu} e^{-i\theta(\Lambda, p)\lambda}|0, \vec{p}_\Lambda, \lambda\rangle}$$

We could check that

- $\mathcal{U}$ is an homomorphism $\mathcal{U}(\Lambda_2, a_2)\,\mathcal{U}(\Lambda_1, a_1) = \mathcal{U}(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$
- $\mathcal{U}$ is unitary $\mathcal{U}^\dagger \mathcal{U} = \mathbb{1}$. *To check it we need to define the scalar product. (the proof is equal to the massive case)*

  - $\langle 0, \vec{p}, \lambda | 0, \vec{q}, \lambda\rangle = (2\pi)^3 (2E_p)\,\delta(\vec{p} - \vec{q})$
  - $\mathbb{1} = \int \frac{d^3\vec{p}}{(2\pi)^3 (2E_p)} |0, \vec{p}, \lambda\rangle\langle 0, \vec{p}, \lambda|$

- $\{|0, \vec{p}, \lambda\rangle\}_\lambda$ *form the minimal invariant subspace under Poincaré* (a P. transf. does not change λ) ⟶ *IRREPS*

☐

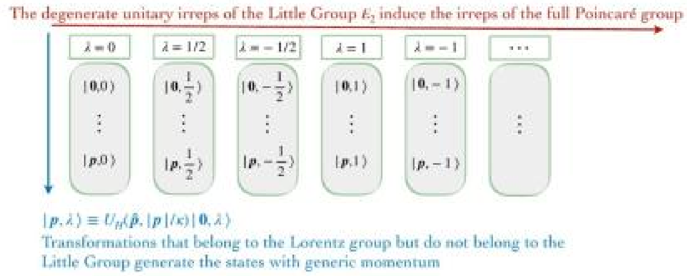


Figura 1.8: *Unitary (infinite-dimensional) irreducible representations of the Poincaré group in the massless case (for ease of reading, we do not show the $M = 0$ label).*

We associate to each unitary irrep of the Poincare group a massless relativistic particle.
We can highlight some intriguing points:

- We found that at the most primitive level, massless particles only have one d.o.f.: their fixed value of helicity.
  However in Q.E.D. we used to think to photons as particle with 2 d.o.f. This is because in Q.E.D. we have an additional space-time symmetry: parity.

  Parity is a L.T. which is not continuously connected to $\mathbb{1}$. This means that the effect of parity is not captured by the action of $U(\Lambda)$. What does the action of parity do to our massless particles?

$$P : (t, \vec{x}) \longmapsto (t, -\vec{x})$$

  Consequently it flips $\vec{P} \to -\vec{P}$ but not $\vec{J}$. and therefore it changes the sign of the helicity. Consequently, in the presence of Parity symmetry the 2 massless states $|p, \pm\lambda\rangle$ are linked and photons appear to have 2 d.o.f.

- Of course in theories without parity symmetry the states with $\pm\lambda$ do not have to be linked. For instance, in the early version of S.M. with massless neutrinos with only $\lambda = -\frac{1}{2}$ and antineutrinos with only $\lambda = \frac{1}{2}$. Weak interactions do not conserve parity and therefore the previous distinctio was legitimate. Nowdays we know that $m_\nu \neq 0$ this means that we need to use the massive representation:

$$\text{massless } \nu : |p, -\tfrac{1}{2}\rangle \ (1 \text{ d.o.f}) \longrightarrow \text{massive } \nu : |m_\nu, p, \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \ (2 \text{ d.o.f.})$$

$$\text{massless } \nu : |p, +\tfrac{1}{2}\rangle \ (1 \text{ d.o.f.}) \longrightarrow \text{massive } \bar{\nu} : |m_{\bar{\nu}}, p, \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \ (2 \text{ d.o.f.})$$

We have another possibility

$$\text{massless } \nu \text{ and } \bar{\nu} : \begin{cases} |p, -\frac{1}{2}\rangle \\ |p, +\frac{1}{2}\rangle \end{cases} \longrightarrow \text{massive } \nu : |m_\nu, p, \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$$

In this last case we pair up the two massless d.o.f. in a single massive state, this is only possible if we declare the ν and the $\bar{\nu}$ to be the same particle. Particles that coincide with their own antiparticles are known as Majorana particles (possible only if they are not charged). Which of the two possibilities is true is still unknown.

- Consider the Wigner phase $\theta(\Lambda, p)$. If we take a pure rotation Λ_R

$$\longrightarrow \boxed{\theta(\Lambda_R(\hat{z}, \phi), p) = \phi}$$

THE VACUUM STATE

Let's consider the last case with $p^2=0$, where $p^0=0$, $\vec{p}=\vec{0}$. First of all since $P^\mu|p^\mu=0\rangle=0$ we can define the state $|0\rangle$ such that:

$$P^\mu|0\rangle=0 \qquad \text{(invariant under space-time translations)}$$

We can now apply the induced representation method:

- The only possible reference vector is: $p_{ref}=(0,0,0,0)$
- Little Group: $G(p)=SO^\uparrow(1,3)$
- The L.G. admits both
  - finite-dim. irreps
    - trivial case: (it does not transforms under Lorentz): unitary $J|0\rangle=0$; $K|0\rangle=0$
    - non-trivial case: we'll encounter in the classification of fields: not unitary
  - ∞-dim. irreps: unitary (much more complicated) $|j_0,\nu,j,m\rangle$

- Those irreps can induce unitary representations of the entire Poincaré group (on the considered orbit $p^2=0$ $p^0=0$ $\vec{p}=0$)

The unitary irreps of the Little Group $SO^\uparrow(1,3)$ induce the irreps of the full Poincaré group on the orbit $p^2=0$ with $p^0=|\mathbf{p}|=0$

Since in this case the Little Group saturates the full Lorentz group, there are no transformations that can be used to generate states with non-vanishing four-momentum.

Figura 1.9: *Unitary (infinite-dimensional) irreducible representations of the Poincaré group in the case $p^2=0$ with $p^0=|\mathbf{p}|=0$. The state $|0\rangle$ is one-dimensional since trivial: it is left invariant by the full Poincaré group.*

- Among all these irreps only the trivial irrep is physical

$$\left.\begin{aligned} P^\mu|0\rangle&=0\\ \vec{K}|0\rangle&=0\\ \vec{J}|0\rangle&=0 \end{aligned}\right\} \longrightarrow \mathcal{U}(\Lambda,a)|0\rangle=0$$

We conclude that we define the vacuum $|0\rangle$ as the state which does not transform under Poincaré.

MULTI-PARTICLE RELATIVISTIC QUANTUM THEORY

It seems we found a relativistic description of a free particle in the massive and massless case. However everything is theoretically flawed : the states we have constructed are eigenstates of definite momentum. while they provide no informations about its position i.e. the particle could be here or in a remote point of the universe.
Therefore this might risk violating principles of causality. To check it we can ask: "What is the amplitude for a particle to travel outside its forward light-cone?

EXERCISE : Particle in a state $|\psi\rangle$ located in $\vec{x}=0, t=0$. $|\psi\rangle \equiv |x=0\rangle$. Let's compute:

$$\langle x| e^{-iHt} |0\rangle \quad \text{with} \quad H = P^0.$$

For simplicity we consider a free scalar particle with mass M and we indicate the momentum eigenstates with $|p\rangle$:

$$H|p\rangle = P^0|p\rangle = E_p|p\rangle$$

If the particle starts from its origin it should be able to make time-like journeys : $\longrightarrow \Delta S^2 > 0$

$$\Delta S^2 = (t-t_0)^2 - |\vec{x}-\vec{x}_0|^2 > 0 \qquad (\text{with } t_0=0,\ \vec{x}_0=0) \qquad \longrightarrow t^2 - |\vec{x}|^2 > 0 \rightarrow -t < |\vec{x}| < t$$

Consequently if we find an amplitude $\neq 0$ for $|\vec{x}| > t$ $\longrightarrow$ the particle can travel faster than light

- NON-RELATIVISTIC QUANTUM MECHANICS: $E_p = \dfrac{|\vec{P}|^2}{2M}$

$$\langle \vec{x}| e^{-iHt} |0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \langle \vec{x}| e^{-iHt} |\vec{p}\rangle\langle p|\vec{x}=0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{-i\frac{|\vec{P}|}{2m}t} \langle \vec{x}|\vec{p}\rangle\langle\vec{p}|\vec{x}=0\rangle =$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{-i\frac{|\vec{P}|^2}{2M}t} e^{i\vec{p}\cdot\vec{x}} = \frac{1}{4\pi^2}\int_0^\infty d|\vec{p}| \int_{-1}^{+1} d\cos\theta\, |\vec{P}|^2 e^{-i\frac{|\vec{P}|t}{2m}} e^{i|\vec{P}||\vec{x}|\cos\theta} =$$

$$= \frac{1}{2\pi^2|\vec{x}|}\int_0^\infty d|\vec{p}|\, |\vec{p}|\, e^{-i\frac{|\vec{P}|^2}{2M}t} \sin(|\vec{p}|\cdot|\vec{x}|) = \left(\frac{M}{2\pi i t}\right)^{3/2} \exp\left(\frac{iM|x|^2}{2t}\right)$$

the result is $\neq 0$ for $\forall x$ and $\forall t$. $\longrightarrow$ NOT POSSIBLE.

- RELATIVISTIC QUANTUM-MECHANICS $E_p = \sqrt{|\vec{p}|^2 + M^2}$

$$\langle \vec{x}| e^{-iHt} |0\rangle = \frac{1}{2\pi^2|\vec{x}|}\int_0^\infty d|\vec{p}|\cdot|\vec{p}|\, e^{-iE_pt} \sin(|\vec{P}|\cdot|\vec{x}|) = \frac{1}{2\pi^2}\int_0^\infty d|\vec{P}||\vec{P}| e^{-iE_pt} \frac{(e^{i|\vec{P}|\cdot|\vec{x}|} - e^{-i|\vec{P}||\vec{x}|})}{2i} =$$

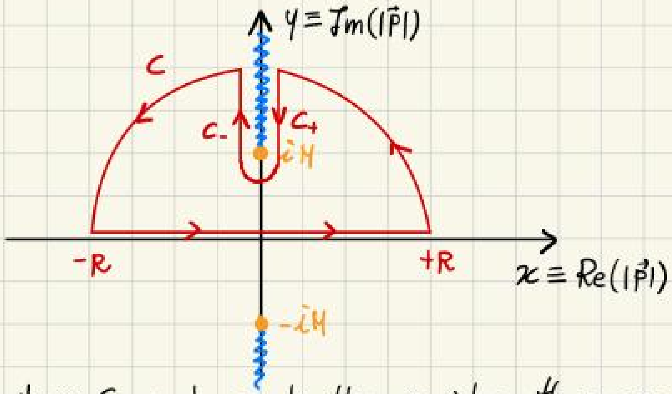
$$= \frac{(-i)}{(2\pi)^2|\vec{x}|}\int_{-\infty}^{+\infty} d|\vec{p}||\vec{p}|\, e^{-it\sqrt{|\vec{P}|^2+m^2} + i|\vec{P}|\cdot|\vec{x}|}$$

We then compute the integral using complex analysis $|\vec{P}| \rightarrow z$

$$f(z) = \frac{(-i)}{(2\pi)^2|\vec{x}|} z\, e^{-it\sqrt{z^2+M^2} + iz|\vec{x}|}$$

$$\longrightarrow 2 \text{ poles } z = \pm iM$$

The presence of $e^{i|P||x|}$ with $|x|>0$ suggests to close the contour in the upper-half of the plane (in order to have an exp. decrease going to large $|P|$ i.e. applying the Jordan lemma).



We integrate over the contour C and apply the residue theorem ⟶ $\oint_C f(z)dz = 0$. In the limit of $R\to\infty$ the integral over the real line reproduces the integral that we want to compute

$$\frac{(-i)}{(2\pi)^2|\vec{x}|}\int_{-\infty}^{+\infty} d|\vec{p}|\,|\vec{p}|\, e^{-it\sqrt{|\vec{p}|^2+M^2}+i|\vec{p}|\cdot|\vec{x}|} + \int_{C_+} f(z)dz + \int_{C_-} f(z)dz + \overset{=0}{\cancel{\int_{\text{arcs}} f(z)dz}} = 0$$

$$\longrightarrow \quad I = -\int_{C_+} f(z)dz - \int_{C_-} f(z)dz = --- = \frac{i}{2\pi^2|\vec{x}|}\int_M^{\infty} dy\, y\, e^{-y|\vec{x}|}\cdot \sinh\left(t\sqrt{y^2-M^2}\right)$$

$$\longrightarrow \boxed{\langle \vec{x}|e^{-iHt}|\vec{0}\rangle = \frac{i}{2\pi^2|\vec{x}|}\int_M^{\infty} dy\, y\, e^{-y|\vec{x}|}\cdot \sinh\left(t\sqrt{y^2-M^2}\right)}$$

The integrand is a product of positive terms and the integral is always $\neq 0$. Let's do an estimation: let's take only the increasing exp. part of sinh.

$$\langle \vec{x}|e^{-iHt}|\vec{0}\rangle < \frac{i}{2\pi^2|\vec{x}|}\int_M^{\infty} dy\, y\, e^{-y(|\vec{x}|-t)} = \frac{i}{2\pi^2|\vec{x}|}\, e^{-M(|\vec{x}|-t)}\left[\frac{1}{(|\vec{x}|-t)^2}+\frac{M}{(|\vec{x}|-t)}\right]$$

Therefore the problem is less dramatic : going outside the light cone is exponentially suppressed:

If $|\vec{x}|-t \gg o(\frac{1}{M})$ ⟶ violation of causality goes to zero fastly

Notice that this problem arises on lenght scales λ that are $\lambda \lesssim o(1/M)$ i.e. we have an uncertainty on the position of the particle of the order $\lesssim o(1/M)$. Therefore, as a consequence of the Heisemberg principle, an uncertainty on the momentum of the order $o(M)$ ⟶ smaller is the spatial confinement greater is its momentum i.e. its energy. This suggests that:

at lenght $\lesssim o(\lambda)$ (and therefore at energies $> o(1/\lambda)$) we have enough energy to have a multi-particle state

Therefore maybe it is not Q.M. that is incompatible with S.R., but rather single-particle Q.M.

We can think that not just Δp is complementary to Δx, but also the # of particles.

Let's go back to the simple non relativistic system of one particle Q.M..

$$H|\vec{p}\rangle = \frac{|\vec{p}|^2}{2m}|\vec{p}\rangle \quad ; \quad P|\vec{p}\rangle = p|\vec{p}\rangle \qquad \langle \vec{p}|\vec{q}\rangle = \delta(\vec{p}-\vec{q})$$

A 2 particle - state which describes 2 independent identical free particles with momenta $\vec{p}_1$ and $\vec{p}_2$ is given by

$$|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle \equiv |\vec{p}_1, \vec{p}_2\rangle$$

of course being identical writing $|\vec{p}_1, \vec{p}_2\rangle$ or $|\vec{p}_2, \vec{p}_1\rangle$ should not matter

$$\longrightarrow |\vec{p}_1 ; \vec{p}_2\rangle \equiv \frac{1}{\sqrt{2}}\left(|\vec{p}_1, \vec{p}_2\rangle + |\vec{p}_2, \vec{p}_1\rangle\right)$$

$$\longrightarrow \quad \langle \vec{p}_1 ; \vec{p}_2 | \vec{p}_1{}' ; \vec{p}_2{}'\rangle = \delta(\vec{p}_1 - \vec{p}_1{}')\,\delta(\vec{p}_2 - \vec{p}_2{}') + \delta(\vec{p}_1 - \vec{p}_2{}')\,\delta(\vec{p}_2 - \vec{p}_1{}')$$

$$H|\vec{p}_1 ; \vec{p}_2\rangle = (E_{p_1} + E_{p_2})|\vec{p}_1 ; \vec{p}_2\rangle \quad ; \quad P|\vec{p}_1 ; \vec{p}_2\rangle = (\vec{p}_1 + \vec{p}_2)|\vec{p}_1 ; \vec{p}_2\rangle$$

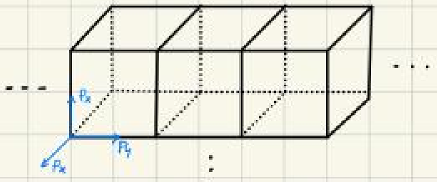
One can proceed with this construction and arrive at building the so called Fock Space i.e. molti-particle Hilbert space. A generic state in the Fock space is:

$$|\psi\rangle = \psi_0|0\rangle + \int d^3\vec{p}\, \psi_1(\vec{p})|\vec{p}\rangle + \frac{1}{2!}\int d^3\vec{p}\, \psi_2(\vec{p}_1, \vec{p}_2)\, |\vec{p}_1 ; \vec{p}_2\rangle + \dots$$

Crazy! It is not very efficient : there's a much better way to do this!

Efficient multi particle theory

We put our system of free independent particles inside a square 3-D. box of length L
We ose periodic bound. cond. which imposes a quantization condition on the possible momentom states that particles in the box can take:



$$\longrightarrow \vec{p} = \left(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L}\right) \qquad n_x, n_y, n_z \in \mathbb{Z}$$

We can describe our states coonting how many particles are there with that momentom : i.e. giving them the occupation number $N(p)$ (# of particles with momentum p)

$$\longrightarrow \boxed{H = \sum_{\vec{p}} E_{\vec{p}} N(\vec{p}) \quad ; \quad \vec{P} = \sum_{\vec{p}} \vec{p}\, N(\vec{p})}$$

for each $\vec{p}$ we sum the quantity $E_{\vec{p}} N(\vec{p})$ = integers multiple of $E_{\vec{p}}$

for each $\vec{p}$ we sum the quantity $\vec{p} \cdot N(\vec{p})$ = integers multiple of $\vec{p}$.

This is precisely what happens in the energy spectrum and momentum spectrum of an harmonic oscillator. *We've built an analogy between 2 different systems : identical particles and a set of uncoupled harmonic oscillators, one for every point in our momentum space - lattice.*

We can extend this analogy further introducing $a^+(\vec{p})$ and $a(\vec{p})$. *In the context of an harmonic oscillator* a^+ and a serve to add or remove quanta from the system. *In our case* a^+ and a are involved in the creation or annihilation of particles in specific momentum states $\longrightarrow$ *multi - particle states will be defined through successive applications of creation operators.*

- We impose in analogy with harmonic oscillator :

$$[a(\vec{p}), a^{\dagger}(\vec{q})] = \delta_{\vec{p}\vec{q}} \qquad [a^{\dagger}(\vec{p}), a^{\dagger}(\vec{q})] = [a(\vec{p}), a(\vec{q})] = 0 \quad \star$$

We can define

$$\boxed{|p_1; p_2; \dots; p_n\rangle \equiv a^{\dagger}(p_1)\, a^{\dagger}(p_2) \cdots a^{\dagger}(p_n)|0\rangle} \qquad \boxed{a(\vec{p})|0\rangle = 0}$$

Due to $\star$ $|p_i; p_j\rangle = |p_j; p_i\rangle$ ⟶ *this formalism describes Bose particles* : we can put any number of them in the same quantum state. Moreover they have integer spin

So, the energy and momentum will be :

$$\boxed{H = \sum_{\vec{p}} E_{\vec{p}}\, a^{\dagger}(\vec{p})a(\vec{p}) \; ; \qquad P = \sum_{\vec{p}} \vec{p}\, a^{\dagger}(\vec{p})a(\vec{p})}$$

- In order to describe fermions we impose :

$$\{c(\vec{p}), c^{\dagger}(\vec{q})\} = \delta_{\vec{p}\vec{q}} \qquad \{c^{\dagger}(\vec{p}), c^{\dagger}(\vec{q})\} = \{c(\vec{p}), c(\vec{q})\} = 0 \quad \star$$

$\star$ prevents that more fermions are in the same quantum state annihilating it. ⟶ so this anticommutation relations allow us to have the Pauli exclusion principle.

We can define

$$\boxed{|\vec{p}_1; \vec{p}_2; \dots; \vec{p}_n\rangle \equiv c^{\dagger}(\vec{p}_1)\, c^{\dagger}(\vec{p}_2) \cdots c^{\dagger}(\vec{p}_n)|0\rangle} \qquad \boxed{c(\vec{p})|0\rangle = 0}$$

Due to $\star$ $|\vec{p}_i; \vec{p}_j\rangle = -|\vec{p}_j; \vec{p}_i\rangle$ as expected.

So, the energy and momentum will be:

$$\boxed{H = \sum_{\vec{p}} E_{\vec{p}}\, c^{\dagger}(\vec{p})c(\vec{p}) \; ; \qquad P = \sum_{\vec{p}} \vec{p}\, c^{\dagger}(\vec{p})c(\vec{p})}$$

We now try to implement all the above considerations in the continuum limit to the states we constructed from the unitary irreps of the Poincaré group.

We indicate with $|n, \vec{p}, \lambda\rangle$ the states that belong to a given irrep of Poincaré group "n" indicates the set of Casimir eigenvalues that define the irrep. e.g. in the massive case $n = \{M, s\}$. Therefore the <u>single particle states</u> are defined by:

$$\boxed{\begin{array}{ll} |n, \vec{p}, \lambda\rangle \equiv \sqrt{2E_p}\, a^{\dagger}(n, \vec{p}, \lambda)|0\rangle & \\ a(n, \vec{p}, \lambda)|0\rangle = 0 & \\ [a(n, \vec{p}, \lambda), a^{\dagger}(n, \vec{p}, \lambda)] = (2\pi)^3 \delta(\vec{p} - \vec{q})\, \delta_{\lambda\lambda'}\delta_{nn'} & \text{integer spin} \\ \{a(n, \vec{p}, \lambda), a^{\dagger}(n', \vec{p}, \lambda')] = (2\pi)^3 \delta(\vec{p} - \vec{q})\, \delta_{\lambda\lambda'}\delta_{nn'} & \text{semi-integer spin} \end{array}}$$

and verify: $\boxed{\langle n', \vec{q} \lambda' | n, \vec{p}, \lambda \rangle = (2\pi)^3 \delta_{nn'} \delta_{\lambda\lambda'} 2 E_p \delta(\vec{p}-\vec{q})}$

Consequently a multi-particle state is defined by:

$$\boxed{|n_1, \vec{p}_1, \lambda_1 ; \dots ; n_N, \vec{p}_N, \lambda_N\rangle \equiv \sqrt{2E_{p_1}}\, a^\dagger(n_1, \vec{p}_1, \lambda_1) \cdots \sqrt{2E_{p_N}}\, a^\dagger(n_N, \vec{p}_N, \lambda_N) |0\rangle}$$

with energy and momentum:

$$\boxed{\begin{aligned} H &= \sum_n \left[\sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3} E_p a^\dagger(n,\vec{p},\lambda) a(n,\vec{p},\lambda) \right] \\ P &= \sum_n \left[\sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p}\, a^\dagger(n,\vec{p},\lambda) a(n,\vec{p},\lambda) \right] \end{aligned}}$$

→ not derived introducing quantum fields but from first principles analyzing the physical d.o.f.

takes into account the possible presence of more than 1 species of particles

sum over the helicities

Let's see that the things work in the case of energy. Let's compute firstly:

$$[H, a^\dagger(n,\vec{p},\lambda)] = \sum_m \sum_{\lambda'} \int \frac{d^3\vec{k}\, E_{\vec{k}}}{(2\pi)^3} [a^\dagger(m,\vec{k},\lambda') a(m,\vec{k},\lambda'), a^\dagger(n,\vec{p},\lambda)]$$

Considering the <u>bosonic case</u> : $[AB,C] = A[B,C] + [A,C]B$

$$= \sum_m \sum_{\lambda'} \int \frac{d^3\vec{k}\, E_{\vec{k}}}{(2\pi)^3} a^\dagger(m,\vec{k},\lambda') [a(m,k,\lambda'), a^\dagger(n,p,\lambda)] =$$

$$= \sum_m \sum_{\lambda'} \int \frac{d^3\vec{k}\, E_{\vec{k}}}{(2\pi)^3} (2\pi)^3 \delta(\vec{p}-\vec{k}) \delta_{\lambda\lambda'} \delta_{nm} = E_p a^\dagger(n,\vec{p},\lambda)$$

Considering the <u>*fermionic case*</u> $[AB,C] = A\{B,C\} - \{A,C\}B$

$$= \sum_m \sum_{\lambda'} \int \frac{d^3\vec{k}\, E_{\vec{k}}}{(2\pi)^3} a^\dagger(m,\vec{k},\lambda') \{a(m,k,\lambda'), a^\dagger(n,p,\lambda)\} = E_p a^\dagger(n,\vec{p},\lambda)$$

Independently from statistics (Bose or Fermi-Dirac) we have:

$$\boxed{[H, a^\dagger(n,\vec{p},\lambda)] = E_p a^\dagger(n,\vec{p},\lambda)}$$

$$\longrightarrow H|n,\vec{p},\lambda\rangle = \sqrt{2E_p}\, H a^\dagger(n,p,\lambda)|0\rangle = \sqrt{2E_p}\left(\cancel{\mp a^\dagger(n,\vec{p},\lambda) H} + E_p a^\dagger(n,\vec{p},\lambda)\right)|0\rangle = E_p |n,\vec{p},\lambda\rangle$$

We can immediately generalize this result to multi-particle states:

$$\boxed{[H, a^\dagger(n_1,\vec{p}_1,\lambda_1) \cdots a^\dagger(n_N,\vec{p}_N,\lambda_N)] = (E_{p_1} + \dots + E_{p_N})\, a^\dagger(n_1,\vec{p}_1,\lambda_1) \cdots a^\dagger(n_N,\vec{p}_N,\lambda_N)}$$

$$\longrightarrow \boxed{H|n_1,\vec{p}_1,\lambda_1; \dots; n_N,\vec{p}_N,\lambda_N\rangle = (E_{p_1} + \dots + E_{p_N})|n_1,\vec{p}_1,\lambda_1; \dots; n_N, p_N, \lambda_N\rangle}$$

We are gradually shifting our focus from states to operators that act upon states (as $a^\dagger$ and a). This maybe will help us to implement causality : we don't localize particles but observations.
Since we know how states transform under a Poincaré transf. it is legitimate to ask about the transformation properties of operators and in particular how $a^\dagger$ and a transform under Poincaré

Massive case :

$$|M,\vec{p},s,\sigma\rangle = \sqrt{2E_p}\, a^\dagger(M,\vec{p},s,\sigma)|0\rangle$$

$$\longrightarrow \qquad \mathcal{U}(\Lambda,a)|M,\vec{p},s,\sigma\rangle = \sqrt{2E_p}\,\mathcal{U}(\Lambda,a)\,a^\dagger(M,\vec{p},s,\sigma)|0\rangle$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$e^{iap_\Lambda}\sum_{\lambda'}\left[D_s(\omega(\Lambda,p))\right]_{\lambda'\lambda}|M,\vec{p}_\Lambda,s,\lambda'\rangle = \sqrt{2E_p}\,\mathcal{U}(\Lambda,a)\,a^\dagger(M,\vec{p},s,\sigma)\,\mathcal{U}(\Lambda,a)^{-1}\underbrace{\mathcal{U}(\Lambda,a)|0\rangle}_{=|0\rangle\ (\text{Vacuum invariant})}$$

$$e^{iap_\Lambda}\sum_{\lambda'}\left[D_s(\omega(\Lambda,p))\right]_{\lambda'\lambda}\sqrt{2E_{p_\Lambda}}\,a^\dagger(M,\vec{p}_\Lambda,s,\lambda')|0\rangle = \sqrt{2E_p}\,\mathcal{U}(\Lambda,a)\,a^\dagger(M,\vec{p},s,\sigma)\,\mathcal{U}(\Lambda,a)^{-1}|0\rangle$$

$$\longrightarrow \quad \boxed{\mathcal{U}(\Lambda,a)\,a^\dagger(M,\vec{p},s,\sigma)\,\mathcal{U}(\Lambda,a)^{-1} = e^{iap_\Lambda}\sqrt{\frac{E_{p_\Lambda}}{E_p}}\sum_{\lambda'}\left[D_s(\omega(\Lambda,p))\right]_{\lambda'\lambda}a^\dagger(M,\vec{p}_\Lambda,s,\lambda')}$$

and in analogy, conjugating it we find:

$$\boxed{\mathcal{U}(\Lambda,a)\,a(M,\vec{p},s,\sigma)\,\mathcal{U}(\Lambda,a)^{-1} = e^{-iap_\Lambda}\sqrt{\frac{E_{p_\Lambda}}{E_p}}\sum_{\lambda'}\left[D_s(\omega(\Lambda,p))\right]^*_{\lambda'\lambda}a(M,\vec{p}_\Lambda,s,\lambda')}$$

Massless case :

$$\boxed{\mathcal{U}(\Lambda,a)\,a^\dagger(\vec{p},\lambda)\,\mathcal{U}(\Lambda,a)^{-1} = e^{ia\cdot p_\Lambda}\sqrt{\frac{E_{p_\Lambda}}{E_p}}\,e^{-i\theta(\Lambda,p)\lambda}a^\dagger(\vec{p}_\Lambda,\lambda)}$$

$$\boxed{\mathcal{U}(\Lambda,a)\,a(\vec{p},\lambda)\,\mathcal{U}(\Lambda,a)^{-1} = e^{-ia\cdot p_\Lambda}\sqrt{\frac{E_{p_\Lambda}}{E_p}}\,e^{i\theta(\Lambda,p)\lambda}a^\dagger(\vec{p}_\Lambda,\lambda)}$$

MASSIVE QUANTUM FIELDS OF ANY SPIN

In Q.M. an hermitian operator is observable, not true in relativistic Q.M.

Implementation of causality in QM: if O_1 corresponds to an observable measured in $X = (\vec{x}, t)$ and O_2 corresponds to an observable measured in $Y = (\vec{y}, t')$

$$\longrightarrow \boxed{[O_1(X), O_2(Y)] = 0 \quad \text{if} \quad (X-Y)^2 < 0}$$

Let's introduce the so called quantum field i.e. the building block of the observables:

$$\boxed{\Phi^a(x) = \Phi^a(\vec{x}, t) \ : \ \text{operator - valued - functions of space - time}}$$

What are the properties of these building block? A quantum field is defined by a set of condition:

1) It's not crucial to be hermitian

2) They satisfy the MICROCAUSALITY condition: $[\Phi^a(x), \Phi^b(y)]_{\pm} = 0 \quad (x-y)^2 < 0$ antic. comm.
With this we have 2 possibilities to implemet relativistic causality:

a) $[\Phi^A(X), \Phi^B(Y)] = 0$ (BOSONIC FIELDS)

In this case $\Phi^A(X)$ are observables. The algebraic properties of commutators that this property is also true for composite operators built using fields as constituent:

$$[A(X)B(X), C(Y)D(Y)] = A\underbrace{[B,C]}_{=0}D + AC\underbrace{[B,D]}_{=0} + \underbrace{[A,C]}_{=0}DB + C\underbrace{[A,D]}_{=0}B = 0$$

b) $\{\Phi_i(X), \Phi_j(Y)\} = 0$ (FERMIONIC FIELDS)

In this case $\Phi^A(X)$ are not observables. However we could construct good observable operators using the product of an even number of quantum fields:

$$[A(X)B(X), C(Y)D(Y)] = A\underbrace{\{B,C\}}_{=0}D + CA\underbrace{\{B,D\}}_{=0} + \underbrace{\{A,C\}}_{=0}DB + C\underbrace{\{A,D\}}_{=0}B = 0$$

3) Transformation properties under Poincaré.
In Q.M. if we have symmetry transformation acting on states, we can reinterpret it as a transformation acting on operators:

$$|\psi\rangle \longrightarrow \mathcal{U}|\psi\rangle \quad \Rightarrow \quad O \longrightarrow O' = \mathcal{U}^+ O \mathcal{U}$$

Key part: What is $\mathcal{U}^+(\Lambda, b)\, \Phi^A(X)\, \mathcal{U}(\Lambda, b) = ?$

$$\boxed{\Phi^A(X) \xrightarrow{(\Lambda, b)} \Phi'^A(X) = \mathcal{U}^+(\Lambda, b)\, \Phi^A(X)\, \mathcal{U}(\Lambda, b) = \overset{\star}{D(\Lambda)}{}^A{}_B\, \Phi^B(\overset{\star}{\Lambda^{-1}X - \Lambda^{-1}b})}$$

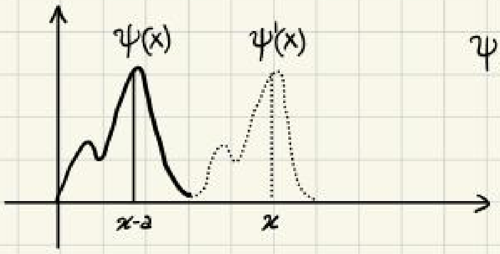
This role in the axiomatic Q.F.T. gives me the def. for a quantum field.

- $D(\Lambda)$ corresponds to some finite dimensional representation of the L.G. according to which $\Phi^A(X)$ transforms. (This representation due to non-compactness of L.G. is not unitary)

MOTIVATION UNDER THE RULE

1) Let's take a scalar function and a transformation:

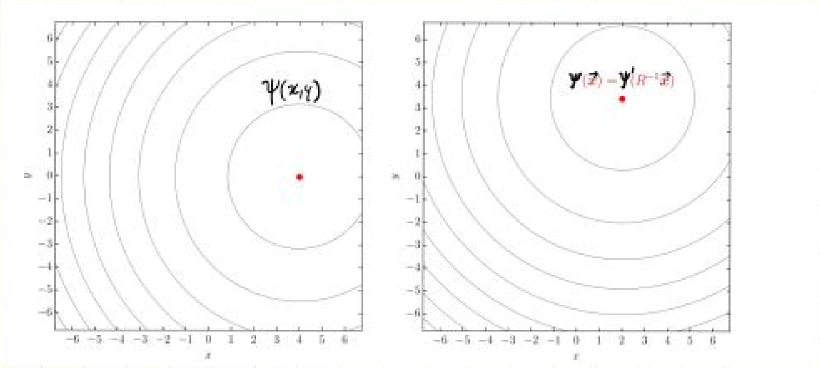
- 1D - translation



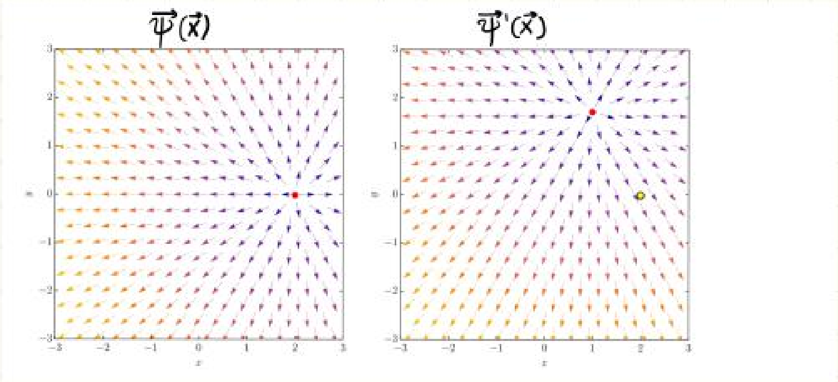
$\psi'(x) = \psi(x-a)$

- 2D - rotation

This explain ★



2) Let's take a vector field , instead of a scalar function:



$\vec{\psi}(\vec{x}) = R\,\vec{\psi}(R^{-1}\vec{x}) = \underbrace{R^A{}_B\,\psi^B}_{\text{this explain} \,\bigstar}\; R^{-1}\vec{X}$

What are the consequences? i.e. What are the relations between generators and Q. FIELDS?

- SPACE-TIME TRANSLATIONS : infinitesimal $x \longrightarrow x+\varepsilon$

$$(\Lambda, b) \longrightarrow (\mathbb{1}, \varepsilon) \qquad \longrightarrow \qquad \mathcal{U}(\mathbb{1}, \varepsilon) = \mathbb{1} + i\,\varepsilon_\mu P^\mu$$

$$\mathcal{U}(\mathbb{1}, \varepsilon)^\dagger \Phi^A(x)\,\mathcal{U}(\mathbb{1}, \varepsilon) = \delta^A{}_B\,\Phi^B(x-\varepsilon)$$

$\downarrow$ expand $\qquad\qquad \downarrow$ expand

P^μ does not change sign (hermitian)

$$(\mathbb{1} - i\,\varepsilon_\mu P^\mu)\;\Phi^A(x)\;(\mathbb{1} + i\,\varepsilon_\mu P^\mu) = \Phi^A(x) - \varepsilon_\mu\,\partial^\mu \Phi^A(x)$$

$$\longrightarrow \quad \Phi^A(x) - i\,\varepsilon_\mu\,[P^\mu, \Phi^A(x)] = \Phi^A(x) - \varepsilon_\mu\,\partial^\mu \Phi^A(x)$$

$$\longrightarrow \quad \boxed{[P^\mu, \Phi^A(x)] = -i\,\partial^\mu \Phi^A(x)} \quad \longrightarrow \quad \begin{cases} [P^0, \Phi^A(x)] = -i\,\partial_t \Phi^A(x) \\ [\vec{P}, \Phi^A(x)] = i\,\vec{\nabla}\Phi^A(x) \end{cases}$$

- LORENTZ TRANSFORMATIONS (infinitesimal)

$$(\Lambda, b) \longrightarrow (\mathbb{1}+\omega, 0) \longrightarrow \mathcal{U}(\mathbb{1}+\omega, 0) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}$$

$$\mathcal{U}(\mathbb{1}+\omega,0)^{\dagger}\Phi^{A}(x)\,\mathcal{U}(\mathbb{1}+\omega,0) = D(\Lambda)^{A}{}_{B}\,\Phi^{B}(\Lambda^{-1}x)$$

$$\left(\mathbb{1}+\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)\Phi^{A}(x)\left(\mathbb{1}-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = D(\Lambda)^{A}{}_{B}\,\Phi^{B}(\Lambda^{-1}x)$$

$$\Phi^{A}(x) + \frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}, \Phi^{A}(x)] + o(\omega^2) = D(\Lambda)^{A}{}_{B}\,\Phi^{B}(\Lambda^{-1}x)$$

- Now remember that we can alway expand the representation in terms of generators (in the finite representation):

$$D(\Lambda)^{A}{}_{B} = \delta^{A}{}_{B} - \frac{i}{2}\omega_{\mu\nu}(J_{s}^{\mu\nu})^{A}{}_{B}$$

- Moreover : $x^{\mu} \longrightarrow x'^{\mu} = \left(\delta^{\mu}{}_{\nu} - \frac{i}{2}\omega_{\rho\sigma}(J_{4vec}^{\rho\sigma})^{\mu}{}_{\nu}\right)x^{\nu} = \ldots = x^{\mu} + \omega^{\mu}{}_{\nu}x^{\nu}$

$\updownarrow$ compact notation

$$(J_{4vec}^{\rho\sigma})^{\mu}{}_{\nu} = i\,(g^{\mu\rho}\delta^{\sigma}{}_{\nu} - g^{\mu\sigma}\delta^{\rho}{}_{\nu})$$

(Actually we're interested in Λ^{-1} : $x^{\mu} \longrightarrow x'^{\mu} = \ldots = x^{\mu} - \omega^{\mu}{}_{\nu}x^{\nu}$)

$$\longrightarrow D(\Lambda)^{A}{}_{B}\,\Phi^{B}(\Lambda^{-1}x) = \left[\delta^{A}{}_{B} - \frac{i}{2}\omega_{\mu\nu}(J_{s}^{\mu\nu})^{A}{}_{B}\right]\Phi^{B}(x^{\mu} - \omega^{\mu}{}_{\nu}x^{\nu}) =$$

Now we expand as follows:

- $\Phi^{B}(x^{\mu} - \omega^{\mu\nu}x_{\nu}) = \Phi^{B}(x) - (\partial^{\mu}\Phi^{b}(x))\,\omega_{\mu\nu}x^{\nu} = \Phi^{b}(x) - \frac{i}{2}\omega_{\mu\nu}[-i(x^{\nu}\partial^{\mu} - x^{\mu}\partial^{\nu})]\Phi^{B}(x) = \Phi^{B}(x) - \frac{i}{2}\omega_{\mu\nu}[i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})]\,\Phi^{B}(x)$

$$\longrightarrow \quad = \left[\delta^{A}{}_{B} - \frac{i}{2}\omega_{\mu\nu}(J_{s}^{\mu\nu})^{A}{}_{B}\right]\left\{\Phi^{B}(x) - \frac{i}{2}\omega_{\mu\nu}[i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})]\,\Phi^{B}(x)\right\} =$$

$$= \Phi^{A}(x) - \frac{i}{2}\omega_{\mu\nu}(J_{s}^{\mu\nu})^{A}{}_{B}\Phi^{B}(x) - \frac{i}{2}\omega_{\mu\nu}[i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})]\Phi^{A}(x)$$

$$= \Phi^{A}(x) - \frac{i}{2}\omega_{\mu\nu}\left[(J^{\mu\nu})^{A}{}_{B} + i\,(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\,\delta^{A}{}_{B}\right]\Phi^{B}(x)$$

$$\longrightarrow \Phi^{A}(x) + \frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}, \Phi^{A}(x)] + o(\omega^2) = \Phi^{A}(x) - \frac{i}{2}\omega_{\mu\nu}\left[(J^{\mu\nu})^{A}{}_{B} + i\,(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\,\delta^{A}{}_{B}\right]\Phi^{B}(x)$$

Comparing the blue terms:

$$\longrightarrow \boxed{[J^{\mu\nu}, \Phi^{A}(x)] = -\left[(J_{s}^{\mu\nu})^{A}{}_{B} + i\,(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\,\delta^{A}{}_{B}\right]\Phi^{B}(x)}$$

It seems appropriate to introduce

$$\boxed{J_{L}^{\mu\nu} =: i\,(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})}$$

and rewrite the relation as follows:

$$\boxed{\left[J^{\mu\nu}, \Phi^A(x)\right] = -\left[(J_S^{\mu\nu})^A{}_B + J_L^{\mu\nu}\,\delta^A{}_B\right]\Phi^B(x)}$$

I restrict the attention to rotations : $J^i = \frac{i}{2}\varepsilon_{ijk}J^{jk}$

$$\longrightarrow \left[J^i, \Phi^A(x)\right] = \frac{1}{2}\varepsilon_{ijk}\left[J^{jk}, \Phi^A(x)\right] = \frac{1}{2}\varepsilon_{ijk}(-1)\left[(J_S^{jk})^A{}_B + J_L^{jk}\,\delta^A{}_B\right]\Phi^B(x) =$$

$$= (-1)\left[\frac{1}{2}\varepsilon_{ijk}(J_s^{jk})^A{}_B + \frac{1}{2}\varepsilon_{ijk}J_L^{jk}\,\delta^A{}_B\right]\Phi^B(x) =$$

$$= (-1)\left[(J_s^i)^A{}_B + J_L^i\,\delta^A{}_B\right]\Phi^B(x) =$$

$$J_L^i = \frac{1}{2}\varepsilon_{ijk}J_L^{jk} = \frac{1}{2}\varepsilon_{iju}\,i\left(x^j\partial^k - x^k\partial^j\right) \underset{\substack{\downarrow \\ \text{antisymmetric} \\ \text{property of } \varepsilon_{ijk}}}{=} \left(\frac{-i}{2}\right)\varepsilon_{ijk}(x^j\partial_k) = \varepsilon_{ijk}x^j(-i)\partial_k$$

in vector form $\longrightarrow \boxed{\vec{J}_L = \vec{x} \times (-i\vec{\nabla})}$

Therefore :

$$\boxed{\left[\vec{J}, \Phi^A(x)\right] = -\left[(\vec{J}_S)^A{}_B + \vec{J}_L\,\delta^A{}_B\right]\Phi^B(x) \quad \text{where} \quad \vec{J}_L = \vec{x}\times(-i\vec{\nabla})}$$

$\vec{J}$: 3 generators of rotations in the unitary representation acting on particle states. It describes the tot. ang. mom.

$\vec{J}_S$: It describes the spin. ang. mom. It arises from the intrinsic transformation property of Φ under Lorentz.

$\vec{J}_L$: It describes the orbital ang. mom. It arises from the space time dependence of Φ

4) Decomposition in terms of a^+ and a

We've seen how quantum fields act on states. We already defined a class of operators of this kind a^+, a. Therefore becomes natural to say that Q. fields are linear combinations of creation and annihilation operators (assumption). Let's write :

$$\boxed{\begin{aligned} &\Phi_+^A(x) = \sum_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} u^A(x,n,\vec{p},\sigma)\,a(n,\vec{p},\sigma) && \text{with} && \mathcal{U}(\Lambda,a)\,\Phi_+^A\,\mathcal{U}(\Lambda,a)^{+} \overset{\bigstar}{=} D_+(\Lambda^{-1})^A{}_B\,\Phi_+^B(\Lambda x+b) \\ &\Phi_-^A(x) = \sum_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} v^A(x,n,\vec{p},\sigma)\,a^+(n,\vec{p},\sigma) && \text{with} && \mathcal{U}(\Lambda,a)\,\Phi_-^A\,\mathcal{U}(\Lambda,a)^{+} = D_-(\Lambda^{-1})^A{}_B\,\Phi_-^B(\Lambda x+b) \end{aligned}}$$

OBSERVATIONS :

- n is an index representing the particle species (particular irrep)
- Φ_+ an Φ_- are free fields (since a^+ and a are defined for a system of free particles)
- Φ_+ and Φ_-, in principle, transform under different $D(\Lambda)$
- u and v are the coefficient functions
- E_p is fixed by on shell relation (e.g. in the massive case $E_p = +\sqrt{|\vec{p}|^2 + M^2}$)

Rewriting of the transformation rule

The transformatio property we wrote is :

$$\mathcal{U}(\Lambda,b)^+\,\Phi^A(x)\,\mathcal{U}(\Lambda,b) = D(\Lambda)^a{}_b\,\Phi^B(\Lambda^{-1}x - \Lambda^{-1}b)$$

we can invert it :

Remember that:

$$\bullet\ \mathcal{U}(\Lambda,b)^\dagger \overset{\overset{\text{unitary}}{\downarrow}}{=} \mathcal{U}(\Lambda,b)^{-1} \overset{\overset{D(g)^{-1}=D(g^{-1})}{\downarrow}}{=} \mathcal{U}(\Lambda^{-1},-\Lambda^{-1}b)$$

$$\bullet\ \mathcal{U}(\Lambda,b) = \mathcal{U}(\Lambda^{-1},-\Lambda^{-1}b)^{-1} = \mathcal{U}(\Lambda^{-1},-\Lambda^{-1}b)^\dagger$$

$$\longrightarrow \mathcal{U}(\Lambda^{-1},-\Lambda^{-1}b)\,\Phi^A\,\mathcal{U}(\Lambda^{-1},-\Lambda^{-1}b)^\dagger = D(\Lambda)^A{}_B\,\Phi^B(\Lambda^{-1}x-\Lambda^{-1}b)$$

Since it is true for any Poincaré transformation I call $\Lambda^{-1}\equiv\tilde{\Lambda}$; $-\Lambda^{-1}b\equiv\tilde{b}$

$$\longrightarrow \boxed{\mathcal{U}(\tilde{\Lambda},\tilde{b})\,\Phi^A(x)\,\mathcal{U}(\tilde{\Lambda},\tilde{b})^\dagger = D(\tilde{\Lambda}^{-1})^A{}_B\,\Phi^B(\tilde{\Lambda}x+b)}$$ (More friendly to use)

The 2 transformation properties in ★ impose a constraint over the coeff. functions u^A and v^A.

- SPACE-TIME TRANSLATION case (we'll only check for v^A)

We know that $\mathcal{U}(\mathbb{1},b)\,\Phi^A_-(x)\,\mathcal{U}(\mathbb{1},b)^\dagger = \delta^A{}_B\,\Phi^B_-(x+b) = \Phi^A_-(x+b)$

$$\longrightarrow \mathcal{U}(\mathbb{1},b)\sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,v^A(x,n,\vec{p},\sigma)\,a^\dagger(n,\vec{p},\sigma)\,\mathcal{U}(\mathbb{1},b)^\dagger = \sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,v^A(x+b,n,\vec{p},\sigma)\,a^\dagger(n,\vec{p},\sigma)$$

We know also that the action of space-time translations is simple:

$$\sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,v^A(x,n,\vec{p},\sigma)\,e^{ip\cdot b}\,a^\dagger(n,\vec{p},\sigma) = \sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,v^A(x+b,n,\vec{p},\sigma)\,a^\dagger(n,\vec{p},\sigma)$$

$$\longrightarrow v^A(x+b,n,\vec{p},\sigma) = v^A(x,n,\vec{p},\sigma)\,e^{ip\cdot b}$$

$$\longrightarrow \boxed{v^A(x,n,\vec{p},\sigma) = e^{ix\cdot p}\,v^A(n,\vec{p},\sigma)}$$

For u^a we have:

$$\boxed{u^a(x,n,\vec{p},\sigma) = e^{-ix\cdot p}\,u^a(n,\vec{p},\sigma)}$$

$$\longrightarrow \boxed{\begin{aligned}\Phi^A_+(x) &= \sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,e^{-ixp}\,u^A(x,n,\vec{p},\sigma)\,a(n,\vec{p},\sigma)\\ \Phi^A_-(x) &= \sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,e^{ixp}\,v^A(x,n,\vec{p},\sigma)\,a^\dagger(n,\vec{p},\sigma)\end{aligned}}\ ★$$

- LORENTZ TRANSFORMATION case

We know that $\mathcal{U}(\Lambda)\,\Phi^A_-\,\mathcal{U}(\Lambda)^\dagger = D_-(\Lambda^{-1})^A{}_B\,\Phi^B(\Lambda x)$. I can plug in ★

$$\boxed{\text{L.H.S.}} \longrightarrow \mathcal{U}(\Lambda)\sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,e^{ixp}\,v^a(x,n,\vec{p},\sigma)\,a^\dagger(n,\vec{p},\sigma)\,\mathcal{U}^\dagger(\Lambda) =$$

$$= \sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,v^a(x,n,\vec{p},\sigma)\,e^{ixp}\left[\mathcal{U}(\Lambda)\,a^\dagger(n,\vec{p},\sigma)\,\mathcal{U}^\dagger(\Lambda)\right]$$

$$= \sum_\sigma\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,v^a(x,n,\vec{p},\sigma)\,e^{ip\cdot x}\sqrt{\frac{E_{p_\Lambda}}{E_p}}\sum_{\sigma'}\left[D_s(\omega_L(\Lambda,p))\right]_{\sigma',\sigma}a^\dagger(n,\vec{p}_\Lambda,\sigma')$$

$$= \sum_{\sigma,\sigma'}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,v^a(x,n,\vec{p},\sigma')\,e^{ipx}\sqrt{\frac{E_{p_\Lambda}}{E_p}}\left[D_s(\omega_L(\Lambda,p))\right]_{\sigma,\sigma'}a^\dagger(n,\vec{p}_\Lambda,\sigma)$$

R.H.S. We change variable in $\vec{p} \rightarrow \vec{p}_\Lambda$ where $p_\Lambda^\mu = \Lambda^\mu{}_\nu p^\nu$ and use the *fact* that $\frac{d^3\vec{p}}{E_p}$ is Lorentz invariant.

$$\longrightarrow \quad D(\Lambda^{-1})^A{}_B \sum_\sigma \int \frac{d^3\vec{p}\,\sqrt{E_p}}{(2\pi)^3\sqrt{2E_p}} v^B(\Lambda x, n, \vec{p}, \sigma)\, e^{ip\cdot\Lambda x} a^\dagger(n, \vec{p}, \sigma) \overset{\text{change of variable}}{=}$$

$$= D(\Lambda^{-1})^A{}_B \sum_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \sqrt{E_{p_\Lambda}}\, v^B(\Lambda x, n, \vec{p}_\Lambda, \sigma)\, e^{ip_\Lambda\cdot\Lambda x} a^\dagger(n, \vec{p}_\Lambda, \sigma) \overset{p_\Lambda\cdot\Lambda x = px}{=}$$

$$= D(\Lambda^{-1})^A{}_B \sum_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \sqrt{\frac{E_{p_\Lambda}}{E_p}}\, e^{ipx}\, v^B(\Lambda x, n, \vec{p}_\Lambda, \sigma)\, a^\dagger(n, \vec{p}_\Lambda, \sigma)$$

$$\longrightarrow \quad \boxed{D(\Lambda^{-1})^A{}_B\, v^B(\Lambda x, n, \vec{p}_\Lambda, \sigma) = \sum_{\sigma'} v^A(x, n, \vec{p}, \sigma') \left[D_s(\omega_L(\Lambda, p))\right]_{\sigma\sigma'}}$$

In analogy we would find :

$$\boxed{D_+(\Lambda^{-1})^A{}_B\, u^B(\Lambda x, n, \vec{p}_\Lambda, \sigma) = \sum_{\sigma'} u^A(x, n, \vec{p}, \sigma') \left[D_s(\omega_L^{-1}(\Lambda, p))\right]_{\sigma'\sigma}}$$

We are ready to impose microcausality. For sure we have that:

$$\left[\Phi_+^A(x), \Phi_+^B(y)\right]_\pm = 0 \quad ; \quad \left[\Phi_-^A(x), \Phi_-^B(y)\right]_\pm = 0$$

But this is not enough we would like to have also that: $\left[\Phi_+^A(x), \Phi_-^B(y)\right]_\pm = 0$. However in general it's $\neq 0$ ⟶ Problem!

Solution : we need to combine Φ_+ and Φ_- in a way that the above condition is satisfied :

$$\longrightarrow \quad \boxed{\Phi^A(x) \equiv \kappa\, \Phi_+^A(x) + \lambda\, \Phi_-^A(x)} \qquad \left(\kappa \text{ and } \lambda \text{ can be found with } \begin{cases} \left[\Phi^A(x), \Phi^B(y)\right]_\pm = 0 \\ \left[\Phi^A(x), \Phi^{\dagger B}(y)\right]_\pm = 0 \end{cases}\right)$$

Consequence : since we're combining Φ_+ and Φ_- into a single object they have to transform under Lorentz in the same way

$$\longrightarrow \quad \boxed{\Phi^A(x) = \sum_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left[\kappa\, u^A(n, \vec{p}, \sigma)\, e^{-ix\cdot p} a(n, \vec{p}, \sigma) + \lambda\, v^A(n, \vec{p}, \sigma)\, a^\dagger(n, \vec{p}, \sigma)\right]}$$

$$\text{with: } \begin{cases} D(\Lambda^{-1})^A{}_B\, v^B(n, \vec{p}_\Lambda, \sigma) = \sum_{\sigma'} v^A(n, \vec{p}, \sigma') \left[D_s(\omega_L(\Lambda, p))\right]_{\sigma\sigma'} \\ D(\Lambda^{-1})^A{}_B\, u^B(n, \vec{p}_\Lambda, \sigma) = \sum_{\sigma'} u^A(n, \vec{p}, \sigma') \left[D_s(\omega_L^{-1}(\Lambda, p))\right]_{\sigma'\sigma} \end{cases}$$

We notice that the last 2 equation can be rewritten in a more convenient form using $D(g)^{-1} = D(g^{-1})$:

$$\boxed{\begin{aligned} \sum_{\sigma'} u^A(n, \vec{p}_\Lambda, \sigma') \left[D_s(\omega_L(\Lambda, p))\right]_{\sigma'\sigma} &= \left(D(\Lambda)\right)^A{}_B\, u^B(n, \vec{p}, \sigma) \\ \sum_{\sigma'} v^A(n, \vec{p}_\Lambda, \sigma') \left[D_s(\omega_L(\Lambda, p))\right]^*_{\sigma\sigma'} &= \left(D(\Lambda)\right)^A{}_B\, v^B(n, \vec{p}, \sigma) \end{aligned}} \quad ★$$

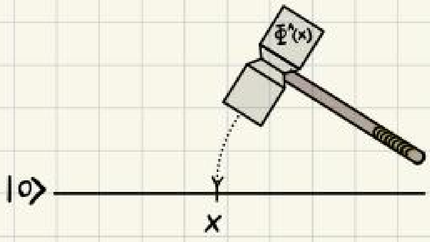
We could do the same for the massless case, obtaining:

$$v^A(\vec{p}_\Lambda, \lambda)\, e^{i\theta(\Lambda,p)\lambda} = D(\Lambda)^A{}_B\, v^B(\vec{p},\lambda)$$
$$u^A(\vec{p}_\Lambda, \lambda)\, e^{-i\theta(\Lambda,p)\lambda} = D(\Lambda)^A{}_B\, u^B(\vec{p},\lambda)$$

Comments:

1) We can apply $\Phi^A(x)$ to the vacuum state:

$$\boxed{\Phi^A(x)\,|0\rangle = \sum_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\, \lambda\, v^A(n,\vec{p},\sigma)\, e^{ixp}\, |n,\vec{p},\sigma\rangle}$$



This is similar to eigenstate of position in Q.M.:

$$|\vec{x}\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\, |\vec{p}\rangle\langle\vec{p}|\vec{x}\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{x}}\, |\vec{p}\rangle$$

Interpretation: we create a particle in space-time position X: the *free field* acts as a sort of hammer which hits the vaccum at position x and shakes single-particle quanta out of it.

2) P^μ and $\Phi^A(x)$ does not commute in general:

$$\boxed{\left[P^\mu, \Phi^A(x)\right] = i\,\partial^\mu\, \Phi^A(x)}\,.$$

We can check it for $\mu = 0$ $\rightarrow P^0 = H = \sum_\sigma \int d^3\vec{p}\; E_p\, a^+ a \longrightarrow \left[H, \Phi^A(x)\right] = i\,\partial^0\, \Phi^A(x)$

3) The Klein-Gordon equation is satisfied (for every component):

$$\boxed{\Box\,\Phi^A(x) + M^2\,\Phi^A(x) = 0}$$

Now we would like to extract infos from the equations ★ (both in massive and massless case).

MASSIVE CASE

1) We consider $\vec{p} = \vec{0}$ (reference vector) and $\Lambda = \Lambda_R$ ($\in SO(3)$)

Consequence: $\begin{cases} \Lambda_R p = p_\Lambda = 0 & \text{(Little Group does not chang } p_{ref}) \\ W_L(\Lambda_R, p) = \Lambda_R \end{cases}$

$$\longrightarrow \boxed{\begin{aligned} D(\hat{n},\theta)^A{}_B\, v^B(n,\vec{0},\sigma) &= \sum_{\sigma'} v^A(n,\vec{0},\sigma')\, \left[D_s(\hat{n},\theta)\right]^*_{\sigma'\sigma} \\ D(\hat{n},\theta)^A{}_B\, u^B(n,\vec{0},\sigma) &= \sum_{\sigma'} u^A(n,\vec{0},\sigma')\, \left[D_s(\hat{n},\theta)\right]_{\sigma'\sigma} \end{aligned}}$$ ★

2) We consider $\vec{p} = \vec{0}$ and $\Lambda = \Lambda_B(\hat{p}, \eta) \equiv L(\hat{p}, \frac{|\vec{p}|}{M})$

<u>Consequence</u>: $\omega_L(\Lambda_B(\hat{p},\eta), p) = \mathbb{1}$

<u>Proof</u>: $\omega_L(\Lambda, p) = L(\hat{p}, \frac{|\vec{p}|}{M})^{-1} \Lambda\, L(0,0) = \Lambda_B(\hat{p},\eta)^{-1} \Lambda_B(\hat{p},\eta) = \mathbb{1}_{4\times4}$ $\quad (\Lambda = \Lambda_B = L(\hat{p}, \frac{|\vec{p}|}{M})$ and $\vec{p} = 0)$

□

Everything simplifies a lot:

$$\boxed{\begin{aligned} u^A(n, \vec{p}, \sigma) &= \left[D(\Lambda_B(\hat{p},\eta))\right]^A{}_B\, u^B(n, \vec{0}, \sigma) \\ v^A(n, \vec{p}, \sigma) &= \left[D(\Lambda_B(\hat{p},\eta))\right]^A{}_B\, v^B(n, \vec{0}, \sigma) \end{aligned}}$$

How to choose the finite representation?

Let's take ★ $\quad D(\hat{n}, \theta)^A{}_B\, u^B(n, \vec{0}, \sigma) = \sum_{\sigma'} u^A(n, \vec{0}, \sigma') \left[D_s(\hat{n}, \theta)\right]_{\sigma'\sigma}$

In order to have chance to solve this equation it is necessary that the finite dim. representation $D(\Lambda)$ when decomposed with the rotation subgroup of the L.G. contains precisely the representation with $j = s$

Example: suppose I want to construct the quantum field corresponding to massive spin 1 particles. We know how the matricies $D_1(\hat{n}, \theta)$ are constructed: (they form the 3-dim repr. of SO(3))

$$\left[D_1(\hat{n}, \theta)\right]_{\sigma'\sigma} = \langle \sigma', 1 | e^{-i \hat{n} \vec{J} \theta} | \sigma, 1 \rangle \text{ where } J^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad J^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J^2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The objects which transform in this way are 3-dim vectors. So the spin 1 quantum field transform under Lorentz in accordance to a representation of L.G. such that, when decomposed under rotations it contains the 3-dim representation of SO(3).

MASSLESS CASE

1) We consider $p = p_{ref}$ and $\Lambda = \Lambda_W \in SE(2)$ (Little Group)

<u>Consequence</u> $p_\Lambda = \Lambda_W p = p_{ref}$

$$\longrightarrow \boxed{u^A(n, \vec{p}_{ref}, \lambda)\, e^{-i\theta(\Lambda_W, p)\lambda} = D(\Lambda_W)^A{}_B\, u^B(n, \vec{p}_{ref}, \lambda)}$$

2) We consider p generic and $\Lambda = H(\hat{p}, \frac{|\vec{p}|}{\kappa})^{-1}$

<u>Consequence</u>
- $p_\Lambda = \Lambda p = H^{-1} p = p_{ref} \longrightarrow p = H(\hat{p}, \frac{|\vec{p}|}{\kappa})\, p_{ref}$ (κ is the comp. of mom. along $\hat{z}$ dir of p_{ref})
- $\omega(\Lambda = H^{-1}, p) = \mathbb{1}_{4\times4} \longrightarrow$ Wigner phase $\theta = 0$ (not rotating)

$$\longrightarrow \boxed{u^A(\vec{p}, \lambda) = D(H^{-1})^A{}_B\, u^B(n, \vec{p}_{ref}, \lambda)}$$

3) We consider $p = p_{ref}$ and $\Lambda = \Lambda_R(\hat{z}, \phi)$

<u>Consequence</u>: $\longrightarrow \theta(\Lambda_R(\hat{z}, \phi), p) = \phi$

finite repr. of L.G. restricted to rot around $\hat{z}$

$$\longrightarrow \boxed{u^A(\vec{p}_{ref}, \lambda) = D(\hat{z}, \phi)^A{}_B\, u^B(\vec{p}_{ref}, \lambda)}$$

FINITE DIMENSIONAL IRREPS OF THE LORENTZ GROUP

We've learned that field components form a finite dimensional (not unitary) representations of the L.G. We've also learned that in order to identify which a field is capable of creating particles with a particular spin it is necessary to understand how the corresponding finite repr. of the L.G. (under which the field transforms) decomposes under rotations. Nature has been kind with us : we only need a basic understanding of angular momentum.

Consider the 6 generators of the Lorentz group $\vec{J}$ and $\vec{K}$: they span the algebra $so(1,3)$
We now define the following complex combinations :

$$\begin{cases} \vec{M} = \dfrac{\vec{J}+i\vec{K}}{2} \\ \vec{N} = \dfrac{\vec{J}-i\vec{K}}{2} \end{cases} \longrightarrow \begin{cases} \vec{J} = \vec{M}+\vec{N} \\ \vec{K} = i\,(\vec{N}-\vec{M}) \end{cases} \longrightarrow \begin{cases} [M^i, M^j] = i\varepsilon_{ijk} M^k \\ [N^i, N^j] = i\varepsilon_{ijk} N^k \\ [M^i, N^j] = 0 \end{cases}$$

Therefore we've reorganized the Lorentz algebra $so(1,3)$ into 2 commuting $su(2)$ algebras:

$$\longrightarrow \quad so(1,3) \cong su(2) \oplus su(2) \quad \Longrightarrow \quad SO^{\uparrow}(1,3) \cong SU(2) \times SU(2)$$

But this is completely wrong because $SO^{\uparrow}(1,3)$ is not compact and $SU(2)\times SU(2)$ it is (since $SU(2)$ is compact)
However we are not working within the context of the real algebra of L.G. but rather within to its complexification

$$\longrightarrow \boxed{so(1,3)_{\mathbb{C}} \cong su(2)_{\mathbb{C}} \oplus su(2)_{\mathbb{C}}}$$

Complex Lie Algebra of L.G.

Formally a representation the group that corresponds to this algebra is given by:

$$D^{(j_1,j_2)}(\Lambda) \equiv D^{(j_1)}(\Lambda) \otimes D^{(j_2)}(\Lambda)$$

that is obtained by exponentiating the Kronecker sum of generators

$$N_{j_1} \oplus_K M_{j_2} \equiv N_{j_1} \otimes \mathbb{1}_{(2j_2+1)\times(2j_2+1)} + \mathbb{1}_{(2j_1+1)\times(2j_1+1)} \otimes M_{j_2}$$

where j_1 and j_2 are label for the irreps of $su(2)_{\mathbb{C}}$

CLASSIFICATION OF L.G. IRREPS

Less rigorously the same problem arises in ordinary non-rel. Q.M. where we have both orbital and spin angular momentum. which obey the rotation group's commutation rules and commute with each other.

STEP 1 : The operators $\{|\vec{M}|^2, M^3, |\vec{N}|^2, N^3\}$ are mutually commuting and must posses a common basis of eigenvectors

$$\boxed{\{|j_1,m_1,j_2,m_2\rangle \equiv |j_1,m_1\rangle \otimes |j_2,m_2\rangle\}_{j_1,j_2} \quad \text{with} \quad -j_1 \le m_1 \le j_1 \;;\; -j_2 \le m_2 \le j_2}$$

equipped with the scalar product $\langle j_1,m_1',j_2,m_2' | j_1,m_1,j_2,m_2\rangle = \langle j_1,m_1' | j_1,m_1\rangle\langle j_2,m_2' | j_2,m_2\rangle = \delta_{m_1'm_1}\delta_{m_2'm_2}$

The operators $|\vec{M}|^2$ and $|\vec{N}|^2$ are Casimirs. Consequently each irreps is labeled by 2 indices (j_1,j_2) fixed

$$\longrightarrow \boxed{D^{(j_1,j_2)}(\Lambda) = D^{(j_1)}(\Lambda) \otimes D^{(j_2)}(\Lambda) \qquad j_1 = 0, \tfrac{1}{2}, 1, \ldots; \quad j_2 = 0, \tfrac{1}{2}, 1, \ldots}$$

* What happens if I restrict to rotations

$$\longrightarrow D^{(j_1,j_2)}(\hat{n},\theta) = e^{-i\theta\hat{n}\cdot\vec{J}} = e^{-i\theta\hat{n}\cdot(\vec{M}+\vec{N})} \overset{\vec{M}\text{ and }\vec{N}\text{ commutes}}{=} e^{-i\theta\hat{n}\cdot\vec{M}} e^{-i\theta\hat{n}\cdot\vec{N}}$$

We write the generic matrix element as:

$$\langle j_1, m_1', j_2, m_2' | D^{(j_1,j_2)}(\hat{n},\theta) | j_1, m_1, j_2, m_2\rangle = \langle j_1, m_1', j_2, m_2' | e^{-i\theta\hat{n}\cdot\vec{N}} \cdot e^{-i\theta\hat{n}\cdot\vec{M}} | j_1, m_1, j_2, m_2\rangle =$$

$$= \sum_{n_1,n_2} \langle j_1, m_1', j_2, m_2' | e^{-i\theta\hat{n}\cdot\vec{M}} | j_1, n_1, j_2, n_2\rangle \langle j_1, n_1, j_2, n_2 | e^{-i\theta\hat{n}\cdot\vec{N}} | j_1, m_1, j_2, m_2\rangle =$$

$$= \sum_{n_1,n_2} \langle j_1, m_1' | e^{-i\theta\hat{n}\cdot\vec{M}} | j_1, n_1\rangle \delta_{m_2' n_2} \cdot \delta_{n_1 m_1} \langle j_2, n_2 | e^{-i\theta\hat{n}\cdot\vec{N}} | j_2, m_2\rangle =$$

$$= \langle j_1, m_1' | e^{-i\theta\hat{n}\cdot\vec{M}} | j_1, m_1\rangle \langle j_2, m_2' | e^{-i\theta\hat{n}\cdot\vec{N}} | j_2, m_2\rangle$$

$$= [D_{j_1}(\hat{n},\theta)]_{m_1' m_1} [D_{j_2}(\hat{n},\theta)]_{m_2' m_2}$$

$$\longrightarrow \boxed{D^{(j_1,j_2)}(\hat{n},\theta) \sim D_{j_1}(\hat{n},\theta) \otimes D_{j_2}(\hat{n},\theta) \quad ; \quad \dim\left[D^{(j_1,j_2)}(\hat{n},\theta)\right] = (2j_1+1)\cdot(2j_2+1)}$$

In other words if we multiply the above 2 matricies according to the written rule we get a new representation of SO(3). However typically $D^{(j_1,j_2)}(\hat{n},\theta)$ is not an irrep of SU(2), even if it is written as product of two irreps.

Therefore the idea is to try to diagonalize simultaneously: $\{|\vec{J}|^2, |\vec{M}|^2, |\vec{N}|^2, J^3\}$

with this choice I can get an irreducible representation. The common eigenbasis is:

$$\boxed{\{|j_1, j_2, j, m\rangle\}_{j_1,j_2} \quad \text{with} \quad |j_1 - j_2| \le j \le j_1 + j_2 \quad \text{and} \quad -j \le m \le j}$$

equipped with: $\langle j_1, j_2, j', m' | j_1, j_2, j, m\rangle = \delta_{jj'}\delta_{mm'}$

The relation between the 2 basis is given by the Clebsh-Gordan coefficients:

$$|j_1, j_2, j, m\rangle = \sum_{m_1,m_2} C^{jm}_{j_1 m_1 j_2 m_2} |j_1, m_1, j_2, m_2\rangle \qquad \text{with} \quad C^{jm}_{j_1 m_1 j_2 m_2} \equiv \langle j_1, m_1, j_2, m_2 | j_1, j_2, j, m\rangle$$

Let's recompute the matrix element:

$$\langle j_1, m_1', j_2, m_2' | D^{(j_1,j_2)}(\hat{n},\theta) | j_1, m_1, j_2, m_2\rangle = \langle j_1, m_1', j_2, m_2' | e^{-i\theta\hat{n}\cdot\vec{J}} | j_1, m_1, j_2, m_2\rangle =$$

$$= \sum_{j,m}\sum_{j',m'} \langle j_1, m_1', j_2, m_2' | j_1, j_2, j, m\rangle \langle j_1, j_2, j, m | e^{-i\theta\hat{n}\cdot\vec{J}} | j_1, j_2, j', m'\rangle \langle j_1, j_2, j', m' | j_1, m_1, j_2, m_2\rangle$$

since $\vec{J}$ does not change the eigenvalue j

(Clebsh-Gordan coeff are equal to their inverse)

$$= \sum_{j,m}\sum_{j',m'} C^{jm}_{j_1 m_1' j_2 m_2'} \; \delta_{jj'} \langle j_1, j_2, j, m | e^{-i\theta\hat{n}\cdot\vec{J}} | j_1, j_2, j, m'\rangle \; C^{j',m'}_{j_1 m_1 j_2 m_2}$$

applico $\delta_{jj'}$

$$\overset{\downarrow}{=} \sum_{j,m}\sum_{m'=-j}^{+j} C^{jm}_{j_1 m_1' j_2 m_2'} C^{j',m'}_{j_1 m_1 j_2 m_2} \langle j_1, j_2, j, m | e^{-i\theta\hat{n}\cdot\vec{J}} | j_1, j_2, j, m'\rangle$$

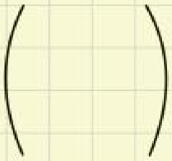
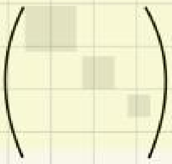
$$= \sum_{j,m}\sum_{m'=-j}^{+j} C^{jm}_{j_1 m_1' j_2 m_2'} C^{j,m'}_{j_1 m_1 j_2 m_2} [D_j(\hat{n},\theta)]_{m,m'}$$

$$\rightarrow \boxed{\langle j_1, m_1', j_2, m_2' \,|\, D^{(j_1,j_2)}(\hat{n},\theta) \,|\, j_1, m_1, j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2}\left[\sum_{m,m'=-j}^{+j} C^{jm}_{j_1 m_1' \, j_2 m_2'} \, C^{j',m'}_{j_1 m_1 \, j_2 m_2} \quad \left[D_j(\hat{n},\theta)\right]_{m,m'}\right]} \bigstar$$

This formula tells us that, under rotations, the irrep $D^{(j_1,j_2)}$ of the full Lorentz is reducible under the subgroup of rotations and it breaks up into the direct sum of $2j+1$ dimensional irreps of the group $SU(2)$ with $|j_1-j_2| \leq j \leq j_1+j_2$

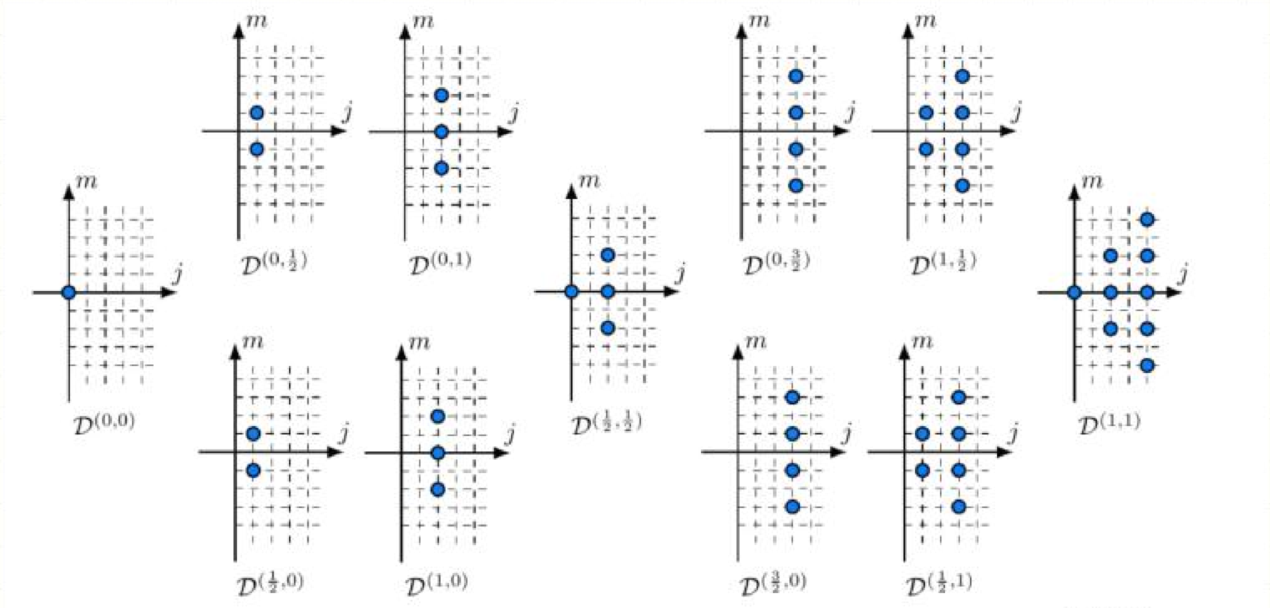
$$\longrightarrow \boxed{D^{(j_1,j_2)}(\hat{n},\theta) \sim D_{j_1}(\hat{n},\theta) \otimes D_{j_2}(\hat{n},\theta) \sim \bigoplus \sum_{j=|j_1-j_2|}^{j_1+j_2} D_j(\hat{n},\theta)}$$

So, graphically we have:

- $\{|j_1, m_1, j_2, m_2\rangle\}$ basis $\longrightarrow$ $D^{(j_1,j_2)}(\hat{n},\theta) =$  $(2j_1+1)\cdot(2j_2+1)$
- $\{|j_1, j_2, j, m\rangle\}$ basis $\longrightarrow$ $D^{(j_1,j_2)}(\hat{n},\theta) =$ 

Therefore we are now able to give a formal answer to the question we posed earlier. If we seek a q. field capable of creating a particle with spin s it is necessary for it to transform according to a *finite-dimensional* repr. of L.G. $D^{(j_1,j_2)}$ such that its corresponding decomposition under rotations includes the representation with $j=s$.

Graphical view of ★



(1.1009)

For example, if we want to describe a scalar particle $s=0$ $\longrightarrow$ we need a repr. with $j=s=0$: $D^{(0,0)}, D^{(1/2,1/2)}, \dots, D^{(j_1,j_2)}$. If we want to describe a $s=1$ particle we need $D^{(1/2,1/2)}, D^{(1,0)}, D^{(0,1)}, \dots$

Okay but how do we make our choice?

* Complex Conjugation

It's useful to consider the properties of complex conjugation. Let's take a generic representation $D(g)$:

$$\begin{aligned} D: G &\longrightarrow GL(V) \\ g &\longmapsto D(g) \end{aligned}$$

We can define the conjugate representation:

$$\begin{aligned} D^*: G &\longrightarrow GL(V) \\ g &\longmapsto D^*(g) \; (\equiv D(g)^*) \end{aligned}$$

The representation is called non-complex representation iff. there exists S such that

$$D^*(g) = S\,D(g)\,S^{-1} \qquad \forall g \in G$$

In the case of a generic Lie group a representation and its conjugate are equivalent (non-complex representation) if there exist a transformation S such that:

$$D(g) = e^{-i\alpha_A T_A} = S\,e^{-i\alpha_A T^A} S^{-1} \longrightarrow S T^A S^{-1} = -(T^A)^*$$

We can show that in the case of $SU(2)$ this is always true for any irreducible representations

Proof (Sketch)

$\vec{M}$ and $\vec{N}$ generate 2 independent $SU(2)$ algebras: $\longrightarrow \begin{cases} S\vec{M}S^{-1} = -\vec{M}^* \\ S\vec{N}S^{-1} = -\vec{N}^* \end{cases}$ therefore $(\vec{J} = \vec{M} + \vec{N}) \rightarrow S\vec{J}S^{-1} = -\vec{J}^*$

The tricky part comes from $\vec{K}$: $\vec{K} = i(\vec{M} - \vec{N})$, in fact:

$$S\vec{K}S^{-1} = i(-\vec{M}^* + \vec{N}^*) = -i(\vec{M}^* - \vec{N}^*) = +\vec{K}^* \longrightarrow \text{Lorentz algebra does not verify } S\cdot S^{-1} = -\cdot^*$$

and therefore the representation is complex (n.b. only for a minus sign)

The representation, however, fails to be complex if we switch the role of $\vec{M} \longleftrightarrow \vec{N}$. Doing this we get indeed a minus sign (which does not change the properly of $S\vec{J}S^{-1} = -\vec{J}^*$)

$$\longrightarrow \left[D^{(j_1,j_2)}(\Lambda)\right]^* \underset{\uparrow}{\sim} D^{(j_2,j_1)}(\Lambda)$$

in general they are complex
with the only exception of
the case where $j_1 = j_2$

$\square$

* Parity

Let's see the effects of a parity transformation on $\vec{J}$ and $\vec{K}$:

$\vec{J} \xmapsto{P} \vec{J}$ (axial-vector / pseudo-vector) $\rightarrow$ because $\begin{cases} \vec{r} \rightarrow -\vec{r} \\ \vec{p} \rightarrow -\vec{p} \end{cases} \rightarrow L = \vec{r} \times \vec{p}$ is invariant and $\vec{S}$ is not affected by spatial inversion $\rightarrow \vec{J} = L + \vec{S}$ is invariant

$\vec{K} \xmapsto{P} -\vec{K}$ (polar-vector / true vector) $\longrightarrow$ because $\vec{v} \longrightarrow -\vec{v}$

Since $\vec{J}=\vec{M}+\vec{N}$ and $\vec{K}=i(\vec{N}-\vec{M})$, a parity transformation switch the role of $\vec{N}$ and $\vec{M}$. Therefore:

$$\boxed{D^{(j_1,j_2)}(\Lambda)\xrightarrow{P}D^{(j_2,j_1)}(\Lambda)}$$

Consequently if the theory regards parity as fondamental symmetry we will have to consider fields transforming according to representations of the type $D^{(j_1,j_2)}$ with $j_1=j_2$ (or simoltaneously incorporate fields transforming according $D^{(j_1,j_2)}$ $j_1\neq j_2$ and fields transforming according $D^{(j_2,j_1)}$).

CURIOSITY: Fields which transform according to $D^{(j_1,j_2)}$ $j_1\neq j_2$ are called : CHIRAL FIELDS

Examples :

- Real scalar field: $\phi(x)\xmapsto{\Lambda}\phi(\Lambda^{-1}x)$: $\phi\sim D^{(0,0)}$
- Real 4-vector field: $V^\mu(x)\xmapsto{\Lambda}\Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x)$: $V^\mu\sim D^{(\frac{1}{2},\frac{1}{2})}$

  Why $D^{(\frac{1}{2},\frac{1}{2})}$? We're looking for a 4-dim representation. In the scheme we have in principle $D^{(0,\frac{3}{2})}, D^{(\frac{3}{2},0)}, D^{(\frac{1}{2},\frac{1}{2})}$. The first 2 can be discarded because: they are not real, but complex repr. instead on the contrary the defining repr $\Lambda^\mu{}_\nu$ is a real repr. ; moreover $D^{(0,\frac{3}{2})}\xmapsto{P}D^{(\frac{3}{2},0)}$ instead a 4 vector may or may not change the spatial component but it certainly remains a 4 vector

- Dirac field: $$\begin{cases}\psi(x)\xmapsto{\Lambda}\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)\\ \Lambda_{\frac{1}{2}}\equiv\exp\left(-\frac{i}{2}\omega_{\mu\nu}J_s^{\mu\nu}\right)\\ J_s^{\mu\nu}=\frac{i}{4}[\gamma^\mu,\gamma^\nu]\end{cases}\qquad \psi\sim D^{(\frac{1}{2},0)}\oplus D^{(0,\frac{1}{2})}$$

  Why $D^{(\frac{1}{2},0)}\oplus D^{(0,\frac{1}{2})}$? Remember that $J_s^i=\frac{1}{2}\varepsilon_{ijk}J_s^{jk}$; $K_s^i=J_s^{0i}$

  There are 2 explicit representations of the Dirac matricies. 2 of the most used are: the Weyl representation and the Dirac representation. We will ose the 1st one:

  Weyl representation $\gamma^0=\begin{pmatrix}0&\mathbb{1}_{2\times2}\\ \mathbb{1}_{2\times2}&0\end{pmatrix}$; $\gamma^i=\begin{pmatrix}0&\sigma^i\\ -\sigma^i&0\end{pmatrix}$; $\gamma^5=\begin{pmatrix}-\mathbb{1}_{2\times2}&0\\ 0&\mathbb{1}_{2\times2}\end{pmatrix}$

  $$\longrightarrow J_s^i=\begin{pmatrix}\frac{\sigma^i}{2}&0\\ 0&\frac{\sigma^i}{2}\end{pmatrix}\qquad K_s^i=\begin{pmatrix}-i\frac{\sigma^i}{2}&0\\ 0&i\frac{\sigma^i}{2}\end{pmatrix}$$

  The block diagonal form of the generators of boost and rotation in the Weyl repr. makes manifest the fact that the representation is reducible

  $$\longrightarrow\psi=\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}\equiv\psi_L+\psi_R\quad;\quad\psi_L\equiv\begin{pmatrix}\psi_L\\ 0\end{pmatrix},\ \psi_R\equiv\begin{pmatrix}0\\ \psi_R\end{pmatrix}$$

  It's easy to see that if we apply a L.T. on ψ_L we'll never generate a ψ_R component and viceversa. I.e. ψ_L and ψ_R represent 2 2-dim invariant subspaces (under the action of $\Lambda_{\frac{1}{2}}$). Each of these 2 components (left and right handed Weyl spinors) transform separately according to irreducible 2-dim repr. of L.G. The block-diagonal form of the generators allow us to write how ψ_L and ψ_R transform:

  $$\psi_L\xmapsto{\Lambda}\exp\left(-i\theta\,\hat{n}\cdot\frac{\vec{\sigma}}{2}-\eta\,\hat{v}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L\qquad\vec{J}=\frac{\vec{\sigma}}{2}\ ;\ \vec{K}=-i\frac{\vec{\sigma}}{2}\ \longrightarrow\ \vec{M}=\frac{\vec{\sigma}}{2}\ ;\ \vec{N}=0\ \longrightarrow\ j_1=\frac{1}{2}\ ;\ j_2=0$$

$$\psi_R \xmapsto{\Lambda} \exp\left(-i\theta\,\hat{n}\cdot\frac{\vec{\sigma}}{2} + \eta\hat{v}\,\frac{\vec{\sigma}}{2}\right)\psi_R \qquad \vec{J}=\frac{\vec{\sigma}}{2}\,;\ \vec{K}=i\frac{\vec{\sigma}}{2} \rightarrow \vec{M}=0\,;\ \vec{N}=\frac{\vec{\sigma}}{2} \rightarrow j_1=0\,;\ j_2=\frac{1}{2}$$

Therefore:

$$\psi_L \sim D^{(\frac{1}{2},0)}\ ;\ \psi_R \sim D^{(0,\frac{1}{2})} \implies \psi \sim D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$$

REAL MASSIVE VECTOR FIELD

The next point is to discuss massive vector fields (important because weak interaction are mediated by spin 1 massive field). N.B. The indeces A,B are replaced with μ,ν.

$$V^\mu(x) = \sum_\sigma \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[\kappa\, u^\mu(\vec{p},\sigma)e^{-ipx}a(\vec{p},\sigma) + \lambda\, v^\mu(\vec{p},\sigma)e^{ipx}a^+(\vec{p},\sigma)\right]$$

We Know yet the 2 master equations:

$$\Lambda_R(\hat{n},\theta)^\mu{}_\nu v^\nu(\vec{0},\sigma) = \sum_{\sigma'} v^A(\vec{0},\sigma')\left[D_s(\hat{n},\theta)\right]^*_{\sigma'\sigma}$$

$$\Lambda_R(\hat{n},\theta)^\mu{}_\nu u^\nu(\vec{0},\sigma) = \sum_{\sigma'} u^A(\vec{0},\sigma')\left[D_s(\hat{n},\theta)\right]_{\sigma'\sigma}$$

Now I can go in the limit of infinitesimal transformation and impose $S=1$:

$$\Lambda_R(\hat{n},\theta) \simeq \mathbb{1} - i\theta\hat{n}\cdot\vec{J}_{4vec}$$

$$D_1(\hat{n},\theta) \simeq \mathbb{1}_{3\times3} - i\theta\hat{n}\cdot\vec{J} \longrightarrow$$ generators that give the representation 3-dim of the rotation group (in the basis $|1,\pm1\rangle, |1,0\rangle$)

$$\longrightarrow \boxed{\begin{aligned}(\vec{J}_{4vec})^\mu{}_\nu u^\nu(\vec{0},\sigma) &= \sum_{\sigma'} u^\mu(\vec{0},\sigma')(\vec{J})_{\sigma'\sigma}\\ (\vec{J}_{4vec})^\mu{}_\nu v^\nu(\vec{0},\sigma) &= -\sum_{\sigma'} v^\mu(\vec{0},\sigma')(\vec{J})^*_{\sigma'\sigma}\ \bigstar\end{aligned}}$$

These are the 2 Key equations we need to solve. Let's do some observations:

1) As we Know the 4-vector repr. of L.G. contains both $j=1$ and $j=0$. We need to isolate the $j=1$ part (sina we are interested in fields which destroy and create spin-1 particles. This is why we set $j=1$ and consequenlly we have the generators of SO(3) in the $S=1$ representation written in the basis $\{|j,m\rangle\}_{j=1}$

2) On the left hand side we have the generators $\vec{J}_{4vec}$ in the defining representation of the L.G. Infact, let's take for example:

$$\cdot\ (\vec{J}_{4vec})^\mu{}_\nu = \left(\begin{array}{c|ccc} 0 & & & \\ \hline & 0 & 0 & 0 \\ & 0 & 0 & -i \\ & 0 & i & 0 \end{array}\right) \longrightarrow \begin{cases}(\vec{J}_{4vec})^0{}_0 = (\vec{J}_{4vec})^0{}_i = (\vec{J}_{4vec})^i{}_0 = 0\\ (J^K_{4vec})^i{}_j = -i\,\varepsilon_{ijk}\end{cases} \longrightarrow (\vec{J}^*_{4vec}) = -\vec{J}_{4vec}$$

We can solve ★ easily setting: (we can check using)

$$\boxed{v^\mu(\vec{0},\sigma) = u^\mu(\vec{0},\sigma)^*}$$

For this reason we now define the so-called polarization vectors at zero momentum:

$$\boxed{\varepsilon^\mu(\vec{0},\sigma) \equiv u^\mu(\vec{0},\sigma) = v^\mu(\vec{0},\sigma)^*}$$

$$\longrightarrow \quad (\vec{J}_{4vec})^{\mu}{}_{\nu}\,\mathcal{E}^{\nu}(\vec{0},\sigma) = \sum_{\sigma'} \mathcal{E}^{\mu}(\vec{0},\sigma')(\vec{J})_{\sigma'\sigma}$$

$$\boxed{\mu=0} \longrightarrow \quad 0 = \sum_{\sigma'} \mathcal{E}^{0}(\vec{0},\sigma')(\vec{J})_{\sigma'\sigma} \longrightarrow \boxed{\mathcal{E}^{0}(\vec{0},\sigma)=0 \quad \text{for all } \sigma}$$

In the rest frame we have $p^{\mu}_{ref} = (M,\vec{0})$. Therefore if we compute: $p^{\mu}_{ref}\,\mathcal{E}_{\mu}(\vec{0},\sigma)$ we find:

$$p^{\mu}_{ref}\,\mathcal{E}_{\mu}(\vec{0},\sigma) = M\cdot\mathcal{E}_{0}(\vec{0},\sigma) = 0$$

Since tha scalar product is a Lorentz invariant quantity this means that this condition must be valid also in the moving frame:

$$\boxed{p_{\mu}\,\mathcal{E}^{\mu}(\vec{p},\sigma) = 0 \qquad \forall\ \sigma=\pm1,0}$$ Trasversality condition

This translates in:

$$\boxed{\partial_{\mu}V^{\mu}(x)=0}$$

We've just found that the field V^{μ}, in order to describe spin-1 particles, must not only satisfy the K.G. equation but also this additional condition. This in practice reduces the # of d.o.f. from 4 to 3.

$$\boxed{\mu=i} \longrightarrow \quad (J^{k}_{4vec})^{i}{}_{j}\,\mathcal{E}^{j}(\vec{0},\sigma) = \sum_{\sigma'}\mathcal{E}^{i}(\vec{0},\sigma')(J^{k})_{\sigma'\sigma}$$

We consider $k=3$:

$$(J^{3}_{4vec})^{i}{}_{j}\,\mathcal{E}^{j}(\vec{0},\sigma) = \sum_{\sigma'}\mathcal{E}^{i}(\vec{0},\sigma')(J^{3})_{\sigma'\sigma}$$

We know that J^3 is diagonal: $J^{3} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \longrightarrow (J^{3})_{\sigma'\sigma} = \sigma\,\delta_{\sigma'\sigma}$

$$\longrightarrow \quad (J^{3}_{4vec})^{i}{}_{j}\,\mathcal{E}^{j}(\vec{0},\sigma) = \sum_{\sigma'}\mathcal{E}^{i}(\vec{0},\sigma')\,\sigma\,\delta_{\sigma'\sigma}$$

$$\longrightarrow \quad \boxed{(J^{3}_{4vec})^{i}{}_{j}\,\mathcal{E}^{j}(\vec{0},\sigma) = \sigma\,\mathcal{E}^{i}(\vec{0},\sigma)}$$ eigenvalue equation for $\vec{\mathcal{E}}(\vec{0},\sigma)$

In order to solve it I can introduce the diagonalisation matrix S:

$$S = \begin{pmatrix}-1/\sqrt{2} & 0 & 1/\sqrt{2}\\ -i/\sqrt{2} & 0 & -i/\sqrt{2}\\ 0 & 1 & 0\end{pmatrix} \longrightarrow S^{-1}J^{3}_{4vec}S = \operatorname{diag}(1,0,-1)$$

$$\longrightarrow \quad S^{-1}J^{3}_{4vec}SS^{-1}\vec{\mathcal{E}}(\vec{0},\sigma) = \sigma S^{-1}\vec{\mathcal{E}}(\vec{0},\sigma)$$

$$\longrightarrow \quad \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\left(S^{-1}\vec{\mathcal{E}}(\vec{0},\sigma)\right) = \sigma\left(S^{-1}\vec{\mathcal{E}}(\vec{0},\sigma)\right)$$

$$\longrightarrow \quad S^{-1}\mathcal{E}^{1} = \begin{pmatrix}1\\0\\0\end{pmatrix} \;;\quad S^{-1}\mathcal{E}^{2} = \begin{pmatrix}0\\1\\0\end{pmatrix} \;;\quad S^{-1}\mathcal{E}^{3} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\longrightarrow \quad \boxed{\vec{\mathcal{E}}(\vec{0},1) = \frac{-1}{\sqrt{2}}\begin{pmatrix}1\\i\\0\end{pmatrix} \;;\quad \vec{\mathcal{E}}(\vec{0},-1) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\end{pmatrix} \;;\quad \vec{\mathcal{E}}(\vec{0},0) = \begin{pmatrix}0\\0\\1\end{pmatrix}}$$

Finally, considering both $\mu=0$, $\mu=i$:

$$\longrightarrow \boxed{\mathcal{E}^{\mu}(0,1)=\frac{-1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix};\quad \mathcal{E}^{\mu}(\vec{0},-1)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix};\quad \mathcal{E}^{\mu}(\vec{0},0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}$$

Now we have to generalize to $\vec{p}\neq\vec{0}$. In order to do that we can apply our friend boost:

$$u^{A}(n,\vec{p},\sigma)=D(\Lambda_B(\hat{p},\eta))^{A}{}_{B}\,u^{B}(n,\vec{0},\sigma)\xrightarrow{\text{rewrite}}u^{\mu}(\vec{p},\sigma)=\underbrace{\Lambda_B(\hat{p},\eta)^{\mu}{}_{\nu}}_{\text{real matrix}}u^{\nu}(\vec{0},\sigma)$$

$$v^{A}(n,\vec{p},\sigma)=D(\Lambda_B(\hat{p},\eta))^{A}{}_{B}\,v^{B}(n,\vec{0},\sigma)\xrightarrow{\text{rewrite}}v^{\mu}(\vec{p},\sigma)=\underbrace{\Lambda_B(\hat{p},\eta)^{\mu}{}_{\nu}}_{\text{real matrix}}v^{\nu}(\vec{0},\sigma)$$

If we take the complex conjugation:

$$u^{\mu}(\vec{p},\sigma)^{*}=\Lambda_B\,u^{\mu}(\vec{0},\sigma)^{*}\longrightarrow\mathcal{E}^{\mu}(\vec{p},\sigma)=\Lambda_B(\hat{p},\eta)^{\mu}{}_{\nu}\,\mathcal{E}^{\nu}(\vec{0},\sigma)$$

The boost matrix is given by (considering a boost along $\hat{z}$):

$$\Lambda_B(\hat{z},\eta)=\begin{pmatrix}E_p/M&0&0&|p|/M\\0&1&0&0\\0&0&1&0\\|p|/M&0&0&E_p/M\end{pmatrix}$$

$$\sigma=0\longrightarrow\mathcal{E}^{\mu}(|p|\hat{z},0)=\Lambda_B(\hat{z},\eta)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}|\vec{p}|/M\\0\\0\\E_p/M\end{pmatrix}$$

LONGITUDINAL POLARIZATION

$$\sigma=-1\longrightarrow\mathcal{E}^{\mu}(|p|\hat{z},+1)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}$$

$$\sigma=+1\longrightarrow\mathcal{E}^{\mu}(|p|\hat{z},+1)=\frac{-1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}$$

TRANSVERSE POLARIZATION
(invariant)

It is intructive to write the polarization vectors in the helicity basis:

$$\mathcal{E}^{\mu}(\vec{p},\lambda)\equiv H\left(\hat{p},\frac{|\vec{p}|}{M}\right)^{\mu}{}_{\nu}\,\mathcal{E}^{\nu}(\vec{0},\sigma)=\Lambda_R(\hat{z}\to\hat{p})^{\mu}{}_{\rho}\,\Lambda_B(\hat{z},\eta)^{\rho}{}_{\nu}\,\mathcal{E}^{\nu}(\vec{0},\sigma)$$

$$\lambda=\pm1\longrightarrow\mathcal{E}^{\mu}(\vec{p},\pm1)=\Lambda_R(\hat{z}\to\hat{p})^{\mu}{}_{\rho}\,\mathcal{E}^{\rho}(|\vec{p}|\hat{z},\pm1)=\frac{e^{\pm i\beta}}{\sqrt{2}}\begin{pmatrix}0\\iS_\beta\mp C_\beta C_\theta\\-iC_\beta\mp S_\beta C_\theta\\\pm S_\theta\end{pmatrix}$$

$$\lambda=0\longrightarrow\mathcal{E}^{\mu}(\vec{p},0)=\Lambda_R(\hat{z}\longrightarrow\hat{p})^{\mu}{}_{\rho}\,\mathcal{E}^{\rho}(|\vec{p}|\hat{z},0)=\begin{pmatrix}|\vec{p}|/M\\\frac{E_p}{M}\hat{p}\end{pmatrix}=\frac{E_p}{M|\vec{p}|}\begin{pmatrix}|\vec{p}|^{2}E_p\\\vec{p}\end{pmatrix}$$

In the ULTRA-RELATIVISTIC LIMIT $E_p=\sqrt{|\vec{p}|^{2}+M^{2}}\simeq|\vec{p}|$

$$\longrightarrow\mathcal{E}^{\mu}(\vec{p},\lambda=0)\simeq\frac{1}{M}\begin{pmatrix}|\vec{p}|\\|\vec{p}|\hat{p}\end{pmatrix}=\frac{p^{\mu}}{M}\longrightarrow\boxed{\mathcal{E}^{\mu}(\vec{p},\lambda=0)\longrightarrow\frac{p^{\mu}}{M}}$$

LONGITUDINAL
POLARIZATION
IN THE ULTRA
RELATIVISTIC LIMIT

important for → massless limit
important for → Higgs mechanism

We, therefore, equally well write the massive vector field in the helicity basis:

$$\boxed{V^\mu(x) = \sum_{\lambda=\pm 1,0} \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left[\kappa\, \varepsilon^\mu(\vec{p},\lambda) e^{-ipx} \cdot a(\vec{p},\lambda) + \eta\, \varepsilon^\mu(\vec{p},\lambda)^* e^{+ip\cdot x} a^\dagger(\vec{p},\lambda) \right]}$$

Exercise: Our goal is the computation of $\sum_{\lambda=\pm 1,0} \varepsilon^\mu(\vec{p},\lambda)^* \varepsilon^\nu(\vec{p},\lambda)$

We remember that $\varepsilon^\mu(\vec{p},\lambda) = \Lambda_R(\hat{z}\to\hat{p})^\mu{}_\nu \varepsilon^\nu(|\vec{p}|\hat{z},\lambda)$. Which it means I can write:

$$\sum_{\lambda=\pm 1,0} \varepsilon^\mu(\vec{p},\lambda)^* \varepsilon^\nu(\vec{p},\lambda) = \Lambda_R(\hat{z}\longrightarrow\hat{p})^\mu{}_\rho\, \Lambda_R(\hat{z}\longrightarrow\hat{p})^\nu{}_\sigma \sum_{\lambda=\pm 1,0} \varepsilon^\rho(\hat{z}|\vec{p}|,\lambda)^* \varepsilon^\sigma(\hat{z}|\vec{p}|,\lambda)$$

Now the computation is rather simple: I know how $\varepsilon^\mu(\hat{z}|\vec{p}|,\lambda)$ are made. Let's take for example: $\lambda=0$

$$\varepsilon^\rho(|\vec{p}|\hat{z},\lambda=0)^* \varepsilon^\sigma(|\vec{p}|\hat{z},\lambda=0) = \begin{pmatrix} \frac{|\vec{p}|^2}{M^2} & 0 & 0 & \frac{|\vec{p}|E_p}{M^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{E_p|\vec{p}|}{M^2} & 0 & 0 & \frac{E_p^2}{M^2} \end{pmatrix} \overset{E_p^2=|\vec{p}|^2+M^2}{=} \begin{pmatrix} \frac{(E_p^2-M^2)}{M^2} & 0 & 0 & \frac{|\vec{p}|E_p}{M^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{E_p|\vec{p}|}{M^2} & 0 & 0 & \frac{|\vec{p}|^2+M^2}{M^2} \end{pmatrix} = \overbrace{\begin{pmatrix} \frac{E_p^2}{M^2} & 0 & 0 & \frac{|\vec{p}|E_p}{M^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{E_p|\vec{p}|}{M^2} & 0 & 0 & \frac{|\vec{p}|^2}{M^2} \end{pmatrix}}^{\star} + \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

I remember that $p_z^\mu = \begin{pmatrix} E_p \\ 0 \\ 0 \\ |\vec{p}| \end{pmatrix}$, therefore I can interpret $\star$ as $\frac{p_z^\mu p_z^\nu}{M^2}$

Now I have to add also the 2 transverse polarizations $(\lambda=\pm 1)$

$$\sum_{\lambda=\pm 1} \varepsilon^\rho(|\vec{p}|\hat{z},\lambda)^* \varepsilon^\sigma(|\vec{p}|\hat{z},\lambda) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore we have

$$\sum_{\lambda=\pm 1,0} \varepsilon^\mu(\hat{z}|\vec{p}|,\lambda)^* \varepsilon^\nu(\hat{z}|\vec{p}|\,\lambda) = \frac{p_z^\mu p_z^\nu}{M^2} + \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{p_z^\mu p_z^\nu}{M^2} - g^{\mu\nu}$$

Finally we can write:

$$\sum_{\lambda=\pm 1,0} \varepsilon^\mu(\vec{p},\lambda)^* \varepsilon^\nu(\vec{p},\lambda) = \Lambda_R(\hat{z}\longrightarrow\hat{p})^\mu{}_\rho\, \Lambda_R(\hat{z}\longrightarrow\hat{p})^\nu{}_\sigma \left(\frac{p_z^\rho p_z^\sigma}{M^2} - g^{\rho\sigma} \right) =$$

$$= \frac{p^\mu p^\nu}{M^2} - g^{\rho\sigma} \Lambda_R{}^\mu{}_\rho \Lambda_R{}^\nu{}_\sigma = \frac{p^\mu p^\nu}{M^2} - g^{\mu\nu}$$

$$\longrightarrow \boxed{\sum_{\lambda=\pm 1,0} \varepsilon^\mu(\vec{p},\lambda)^* \varepsilon^\nu(\vec{p},\lambda) = \frac{p^\mu p^\nu}{M^2} - g^{\mu\nu}}$$

We now have to check the microcausality condition :

$$\left[V^{\mu}(x), V^{\nu}(y)\right]=0 \quad \left[V^{\mu}(x), V^{\nu}(y)^{+}\right]=0$$

We've seen the correlation between spin statistics and commutation / anticommutation relations. We are describing spin-1 particles i.e. bosons , therefore we have commutation relations (instead of anticommutators). However let's consider the possibility that thes 2 q fields can commut or anticommute , just for challenging the spin - statistic theorem .

$$\left[V^{\mu}(x), V^{\nu}(y)\right]_{\pm} = \sum_{\lambda,\lambda'} \int \frac{d^3\vec{p}\, d^3\vec{k}}{(2\pi)^3\sqrt{2E_p}\,(2\pi)^3\sqrt{2E_k}} \Big[\kappa\, \varepsilon^{\mu}(\vec{p},\lambda) e^{-ip\cdot x} a(\vec{p},\lambda) + \eta\, \varepsilon^{\mu}(\vec{p},\lambda)^{*} e^{ip\cdot x} a^{+}(\vec{p},\lambda) ,$$

$$, \kappa\, \varepsilon^{\nu}(\vec{k},\lambda') e^{-ik\cdot y} a(\vec{k},\lambda') + \eta\, \varepsilon^{\nu}(\vec{k},\lambda')^{*} e^{ik\cdot y} a^{+}(\vec{k},\lambda')\Big] =$$

$$= \sum_{\lambda,\lambda'} \int \frac{d^3\vec{p}\, d^3\vec{k}}{(2\pi)^3\sqrt{2E_p}\,(2\pi)^3\sqrt{2E_k}} \Big[\eta\kappa\, \varepsilon^{\mu}(\vec{p},\lambda)\cdot\varepsilon^{\nu}(\vec{k},\lambda')^{*}\cdot e^{-ipx+iky} \left[a(\vec{p},\lambda), a^{+}(\vec{k},\lambda')\right]_{\pm} \pm$$

$$\pm\, \kappa\eta\, \varepsilon^{\mu}(\vec{p},\lambda)^{*} \varepsilon^{\nu}(\vec{k},\lambda')\, e^{ipx-iky} \left[a^{+}(\vec{p},\lambda), a(\vec{k},\lambda')\right]_{\pm} =$$

$$= \sum_{\lambda,\lambda'} \int \frac{d^3\vec{p}\, d^3\vec{k}}{(2\pi)^3\sqrt{2E_p}\,(2\pi)^3\sqrt{2E_k}} \Big[\eta\kappa\, \varepsilon^{\mu}(\vec{p},\lambda)\cdot\varepsilon^{\nu}(\vec{k},\lambda')^{*}\cdot e^{-ipx+iky} (2\pi)^3 \delta_{\lambda\lambda'}\, \delta^3(\vec{p}-\vec{k}) \pm$$

$$\pm\, \kappa\eta\, \varepsilon^{\mu}(\vec{p},\lambda)^{*} \varepsilon^{\nu}(\vec{k},\lambda')\, e^{ipx-iky} (2\pi)^3 \delta_{\lambda\lambda'}\, \delta^3(\vec{p}-\vec{k}) =$$

$$= \sum_{\lambda,\lambda'} \int \frac{d^3\vec{p}\, d^3\vec{k}}{\cancel{(2\pi)^3}\sqrt{2E_p}\,(2\pi)^3\sqrt{2E_k}} \cancel{(2\pi)^3} \delta_{\lambda\lambda'}\, \delta^3(\vec{p}-\vec{k}) \Big[\eta\kappa\, \varepsilon^{\mu}(\vec{p},\lambda)\cdot\varepsilon^{\nu}(\vec{k},\lambda')^{*}\cdot e^{-ipx+iky} \pm \kappa\eta\, \varepsilon^{\mu}(\vec{p},\lambda)^{*} \varepsilon^{\nu}(\vec{k},\lambda')\, e^{ipx-iky}\Big] =$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \cdot \kappa\eta \sum_{\lambda} \varepsilon^{\mu}(\vec{p},\lambda)\varepsilon^{\nu}(\vec{p},\lambda)^{*} \left(e^{-ip\cdot(x-y)} \pm e^{ip(x-y)}\right) =$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \cdot \kappa\eta \left(-g^{\mu\nu} + \frac{p^{\mu}p^{\nu}}{M^2}\right)\left(e^{-ip\cdot(x-y)} \pm e^{ip(x-y)}\right)$$

Trick: I can use the following identity

$$\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}} e^{\mp ip(x-y)} = \frac{\partial}{\partial x^{\nu}}(\mp i p_{\mu}) e^{\mp ip(x-y)} = (\mp i p_{\nu})(\mp i p_{\mu})\, e^{\mp ip(x-y)} = -p_{\mu}p_{\nu}\, e^{\mp ip(x-y)}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \cdot \kappa\eta \left(-g^{\mu\nu} - \frac{\partial_x^{\mu}\partial_x^{\nu}}{M^2}\right)\left(e^{-ip\cdot(x-y)} \pm e^{ip(x-y)}\right) =$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} \cdot \kappa\eta \left(-g^{\mu\nu} - \frac{\partial_x^{\mu}\partial_x^{\nu}}{M^2}\right)\left(e^{-ip\cdot(x-y)} \pm e^{-ip(y-x)}\right) =$$

Now I could define:

$$\boxed{\Delta_{+}(z, M^2) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p} e^{-ip\cdot z}}$$

$$= \kappa\eta \left(-g^{\mu\nu} - \frac{\partial_x^{\mu}\partial_x^{\nu}}{M^2}\right)\left(\Delta_{+}(x-y) \pm \Delta_{+}(y-x)\right)$$

Theorem : $\Delta_+(z)$ is invariant under $O^\uparrow(1,3)$: $\Delta_+(\Lambda z) = \Delta_+(z) \quad \forall \Lambda \in O^\uparrow(1,3)$

Corollary : If $(x-y)^2 < 0 \longrightarrow \Delta_+(x-y) = \Delta_+(y-x)$

If $(x-y)^2 < 0 \longrightarrow$
$$= \kappa\eta\left(-g^{\mu\nu} - \frac{\partial_x^\mu \partial_x^\nu}{M^2}\right)\left(\Delta_+(x-y) \pm \Delta_+(x-y)\right)$$

Therefore if I consider the + sign it is in general different from zero. (anticomm. rel.)
I must consider the − sign (commutation relations)

$$= 0$$

This gives us the so called spin statistic theorem : relation between commutation relation and Bose statistics and between anticommutation relations and Fermi statistics.

Let's check the 2nd condition :

$$\left[V^\mu(x), V^\nu(y)^\dagger\right] = \sum_{\lambda,\lambda'} \int \frac{d^3\vec{p}\, d^3\vec{k}}{(2\pi)^3\sqrt{2E_p}(2\pi)^3\sqrt{2E_k}} \Big[\kappa\, \varepsilon^\mu(\vec{p},\lambda) e^{-ip\cdot x} a(\vec{p},\lambda) + \eta\, \varepsilon^\mu(\vec{p},\lambda)^* e^{ip\cdot x} a^\dagger(\vec{p},\lambda)\, ,$$
$$, \kappa^* \varepsilon^\nu(\vec{k},\lambda')^* e^{+ik\cdot y} a^\dagger(\vec{k},\lambda') + \eta^* \varepsilon^\nu(\vec{k},\lambda')\, e^{-ik\cdot y} a(\vec{k},\lambda')\Big] =$$

$$= \sum_{\lambda,\lambda'} \int \frac{d^3\vec{p}\, d^3\vec{k}}{(2\pi)^3\sqrt{2E_p}(2\pi)^3\sqrt{2E_k}} \Big[|\kappa|^2 \varepsilon^\mu(\vec{p},\lambda)\cdot\varepsilon^\nu(\vec{k},\lambda')^* \cdot e^{-ipx+iky}\left[a(\vec{p},\lambda), a^\dagger(\vec{k},\lambda')\right] +$$
$$-|\eta|^2 \varepsilon^\mu(\vec{p},\lambda)^* \varepsilon^\nu(\vec{k},\lambda')\, e^{ipx-iky}\left[a^\dagger(\vec{p},\lambda), a(\vec{k},\lambda')\right] =$$

$$= \sum_{\lambda,\lambda'} \int \frac{d^3\vec{p}\, d^3\vec{k}}{(2\pi)^3\sqrt{2E_p}(2\pi)^3\sqrt{2E_k}} \Big[|\kappa|^2 \varepsilon^\mu(\vec{p},\lambda)\cdot\varepsilon^\nu(\vec{k},\lambda')^* \cdot e^{-ipx+iky} (2\pi)^3 \delta_{\lambda\lambda'}\, \delta^3(\vec{p}-\vec{k}) +$$
$$-|\eta|^2 \varepsilon^\mu(\vec{p},\lambda)^* \varepsilon^\nu(\vec{k},\lambda')\, e^{ipx-iky} (2\pi)^3 \delta_{\lambda\lambda'}\, \delta^3(\vec{p}-\vec{k}) =$$

$$= \sum_{\lambda,\lambda'} \int \frac{d^3\vec{p}\, d^3\vec{k}}{\cancel{(2\pi)^3}\sqrt{2E_p}(2\pi)^3\sqrt{2E_k}} \cancel{(2\pi)^3} \delta_{\lambda\lambda'}\, \delta^3(\vec{p}-\vec{k}) \Big[|\kappa|^2 \varepsilon^\mu(\vec{p},\lambda)\cdot\varepsilon^\nu(\vec{k},\lambda')^* \cdot e^{-ipx+iky} - |\eta|^2 \varepsilon^\mu(\vec{p},\lambda)^* \varepsilon^\nu(\vec{k},\lambda')\, e^{ipx-iky}\Big] =$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \cdot \sum_\lambda \varepsilon^\mu(\vec{p},\lambda)\varepsilon^\nu(\vec{p},\lambda)^* \left(|\kappa|^2 e^{-ip\cdot(x-y)} - |\eta|^2 e^{ip(x-y)}\right) =$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}\right)\left(|\kappa|^2 e^{-ip\cdot(x-y)} - |\eta|^2 e^{ip(x-y)}\right) =$$

$$= \left(-g^{\mu\nu} - \frac{\partial_x^\mu \partial_x^\nu}{M^2}\right)\left(|\kappa|^2 \Delta_+(x-y) - |\eta|^2 \Delta_+(x-y)\right) =$$

$$= \left(-g^{\mu\nu} - \frac{\partial_x^\mu \partial_x^\nu}{M^2}\right)\left(|\kappa|^2 - |\eta|^2\right)\Delta_+(x-y) = 0 \quad \longleftrightarrow \quad |\kappa|^2 = |\eta|^2$$

We could write $\kappa = |\kappa| e^{i\theta_\kappa}$; $\eta = |\kappa| e^{i\theta_\eta}$

$$\longrightarrow V^\mu(x) = |\kappa| \cdot \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[e^{i\theta_\kappa} \varepsilon^\mu(\vec{p},\lambda) e^{-ipx} a(\vec{p},\lambda) + e^{i\theta_\eta} \varepsilon^\mu(\vec{p},\lambda) e^{ipx} a^\dagger(\vec{p},\lambda)\right]$$

Now I can redefine:

$$\begin{cases} a(\vec{p},\lambda) \longrightarrow e^{i\theta} a(\vec{p},\lambda) \\ a^\dagger(\vec{p},\lambda) \longrightarrow e^{-i\theta} a^\dagger(\vec{p},\lambda) \end{cases} \qquad \text{where} \quad \theta = \frac{\theta_\eta - \theta_\kappa}{2}$$

$$\longrightarrow \quad V^\mu(x) = |\kappa| \cdot e^{\frac{i(\theta_\kappa+\theta_\eta)}{2}} \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left[\varepsilon^\mu(\vec{p},\lambda) e^{-ipx} a(\vec{p},\lambda) + \varepsilon^\mu(\vec{p},\lambda) e^{ipx} a^\dagger(\vec{p},\lambda) \right]$$

The Physics is invariant under rescaling of a. fields. This part of freedom that we have is called field redefinition and it will be important in renormalization theory.

$$\boxed{\text{I can set } |\kappa| \cdot e^{\frac{i(\theta_\kappa+\theta_\eta)}{2}} = 1}$$

So we finally arrived at what is known as CAUSAL VECTOR FIELD:

$$\boxed{V^\mu(x) = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left[\varepsilon^\mu(\vec{p},\lambda) e^{-ipx} a(\vec{p},\lambda) + \varepsilon^\mu(\vec{p},\lambda)^* e^{ipx} a^\dagger(\vec{p},\lambda) \right]}$$

<u>OBSERVATION</u>: How the polarization vectors transform under Lorentz transformations: ☆

$$\Lambda^\mu{}_\nu \varepsilon^\nu(\vec{p},\lambda) = \sum_{\sigma'} \varepsilon^\mu(\vec{p}_\Lambda,\sigma') \left[D_s(\omega_L(\Lambda,p)) \right]_{\sigma'\sigma} \qquad \text{(Spin basis)}$$

$$\Lambda^\mu{}_\nu \varepsilon^\nu(\vec{p},\lambda) = \sum_{\lambda'} \varepsilon^\mu(\vec{p}_\Lambda,\lambda') \left[D_s(\omega_L(\Lambda,p)) \right]_{\lambda'\lambda} \qquad \text{(Helicity basis)}$$

We observe that the polarization vectors transform under Lorentz exactly as the physical states they aim to describe.

$$\mathcal{U}(\Lambda) |M,\vec{p},s,\sigma\rangle = \sum_{\sigma'} \left[D_s(\omega(\Lambda,p)) \right]_{\sigma'\sigma} |M,\vec{p},s,\sigma'\rangle$$

$$\mathcal{U}(\Lambda) |M,\vec{p},s,\lambda\rangle = \sum_{\lambda'} \left[D_s(\omega(\Lambda,p)) \right]_{\lambda'\lambda} |M,\vec{p},s,\lambda'\rangle$$

Natorally they do so according to a different repr.: the pol. vectors following the defining repr. of L.G. (as they transform as 4-vectors) while the physical states following the unitary transformation implemented by $\mathcal{U}(\Lambda)$. The effect is the same!

FEYNMAN PROPAGATOR

I want to compute:

$$\langle 0 | T[V^\mu(x) V^\nu(y)] | 0 \rangle \qquad \text{where} \quad T[V^\mu(x) V^\nu(y)] = \begin{cases} V^\mu(x) V^\nu(y) & , x^0 > y^0 \\ V^\nu(y) V^\mu(x) & , x^0 < y^0 \end{cases}$$

$$\longrightarrow \langle 0 | T[V^\mu(x) V^\nu(y)] | 0 \rangle = \theta(x^0 - y^0) \langle 0 | V^\mu(x) V^\nu(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | V^\nu(y) V^\mu(x) | 0 \rangle$$

We know that:

$$[V^\mu(x), V^\nu(y)] = \left(-g^{\mu\nu} - \frac{1}{M^2}\partial_x^\mu \partial_x^\nu\right)\left[\Delta_+(x-y) - \Delta_+(y-x)\right]$$

I want to relate 🟠 and 🟢

- $\langle 0 | V^\mu(x)\, V^\nu(y) | 0 \rangle = \left(-g^{\mu\nu} - \frac{1}{M^2}\partial_x^\mu \partial_x^\nu\right) \Delta_+(x-y)$
- $\langle 0 | V^\nu(y)\, V^\mu(x) | 0 \rangle = \left(-g^{\mu\nu} - \frac{1}{M^2}\partial_x^\mu \partial_x^\nu\right) \Delta_+(y-x)$

Therefore:

$$\langle 0 | T[V^\mu(x) V^\nu(y)] | 0 \rangle \overset{?}{=} \left(-g^{\mu\nu} - \frac{1}{M^2}\partial_x^\mu \partial_x^\nu\right) \cdot \left[\theta(x^0-y^0)\,\Delta_+(x-y) + \theta(y^0-x^0)\,\Delta_+(y-x)\right]$$

This procedure must be checked because the time derivatives will also act on the θs when we attempt to move them to left. Let's see case by case:

$\mu = i,\ \nu = j$: no problem because θ depends only on space components

$\mu = 0,\ \nu = i$:

$$\theta(x^0-y^0)\,\partial_x^0 \partial_x^i\, \Delta_x(x-y) + \theta(y^0-x^0)\,\partial_x^0 \partial_x^i\, \Delta_+(y-x) =$$

$$= \overset{=0}{\underbrace{\cancel{\partial_x^0\left[\theta(x^0-y^0)\,\partial_x^i \Delta_+(x-y)\right]}}_{\text{tot. der.}}} - \delta(x^0-y^0)\,\partial_x^i \Delta_+(x-y) + \overset{=0}{\underbrace{\cancel{\partial_x^0\left[\theta(y^0-x^0)\,\partial_x^i \Delta_+(y-x)\right]}}_{\text{tot. der.}}} - \delta(x^0-y^0)\,\partial_x^i\, \Delta_+(y-x) = 0$$

Since $\delta(x^0-y^0)$ force to have $x^0 = y^0 \longrightarrow \Delta_+(x-y) = 0$

$\mu = 0,\ \nu = 0$ (where the problem arises)

The extra term that we get from this case is: (neglecting tot. derivatives)

$$+\frac{1}{M^2}\left[\partial_0^x \theta(x^0-y^0)\right]\partial_0^x \Delta_+(x-y) + \frac{1}{M^2}\left[\partial_0^x \theta(y^0-x^0)\right]\partial_0^x \Delta_+(y-x) =$$

$$= +\frac{1}{M^2}\left[\delta(x^0-y^0)\right]\partial_0^x \Delta_+(x-y) - \frac{1}{M^2}\left[\delta(y^0-x^0)\right]\partial_0^x\, \Delta_+(y-x)$$

We can write:

$$\delta(x^0-y^0)\,\partial_0^x\, \Delta_+(x-y) = \delta(x^0-y^0)\,\partial_0^x \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p}\, e^{-iE_p(x^0-y^0) + i\vec{p}\cdot(\vec{x}-\vec{y})} =$$

$$= \frac{-i}{2}\,\delta(x^0-y^0) \int \frac{d^3\vec{p}}{(2\pi)^3\, \cancel{E_p}} (\cancel{E_p})\, e^{-iE_p(x^0-y^0) + i\vec{p}\cdot(\vec{x}-\vec{y})} =$$

$$= \frac{-i}{2} \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = -\frac{i}{2}\,\delta(x^0-y^0)\,\delta^3(\vec{x}-\vec{y}) = \frac{-i}{2}\,\delta^4(x-y)$$

$$\delta(x^0-y^0)\,\partial_0^x\, \Delta_+(x-y) = \delta(x^0-y^0)\,\partial_0^x \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_p}\, e^{iE_p(x^0-y^0) - i\vec{p}\cdot(\vec{x}-\vec{y})} =$$

$$= \frac{i}{2}\,\delta(x^0-y^0) \int \frac{d^3\vec{p}}{(2\pi)^3\, \cancel{E_p}} (\cancel{E_p})\, e^{iE_p(x^0-y^0) - i\vec{p}\cdot(\vec{x}-\vec{y})} =$$

$$= \frac{i}{2} \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} = \frac{i}{2}\,\delta(x^0-y^0)\,\delta^3(\vec{x}-\vec{y}) = \frac{i}{2}\,\delta^4(x-y)$$

So in conclusion the extra term that we get is

$$-\frac{i}{M^2}\delta^4(x-y)$$

$$\longrightarrow \langle 0|T[V^\mu(x)V^\nu(y)]|0\rangle = \left(-g^{\mu\nu}-\frac{1}{M^2}\partial_x^\mu\partial_x^\nu\right)\cdot\left[\theta(x^0-y^0)\,\Delta_+(x-y)+\theta(y^0-x^0)\,\Delta_+(y-x)\right] - \frac{i}{M^2}\delta(x-y)g^{\mu 0}g^{\nu 0}$$

Feynman propagator in position space : $\lim_{\varepsilon\to 0^+}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{p^2-M^2+i\varepsilon}$

$$\boxed{\longrightarrow \langle 0|T[V^\mu(x)V^\nu(y)]|0\rangle = \lim_{\varepsilon\to 0^+}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{M^2}\right)\frac{i}{p^2-M^2+i\varepsilon} \quad -\frac{i}{M^2}g^{\mu 0}g^{\nu 0}\delta^4(x-y)}$$

The extra term called *contact interaction term* seems to destroy the Lorentz covariance of the theory. However we anticipate that it can simply be dropped when one performs perturbative calculations using the Feynman propagator. For this reason we can neglegt it and define the so called Feynman propagator in the position space and momentum space as:

$$\boxed{G_F^{\mu\nu}(x-y) \equiv \lim_{\varepsilon\to 0^+}\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\left(-g_{\mu\nu}+\frac{p_\mu p_\nu}{M^2}\right)\frac{i}{p^2-M^2+i\varepsilon}}$$

position space

$$\boxed{G_F^{\mu\nu}(p) \equiv \left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{M^2}\right)\frac{i}{p^2-M^2+i\varepsilon}}$$

momentum space

REAL MASSLESS VECTOR FIELD (at the origin of Gauge redondancy)

We know very well that $\lambda = \pm 1$ (only transversal d.o.f. are physical). We introduce the following vector field:

$$A^\mu(x) = \sum_{\lambda=\pm 1}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[\kappa\, u^\mu(\vec{p},\lambda)e^{-ip\cdot x}a(\vec{p},\lambda) + \eta\, v^\mu(\vec{p},\lambda)e^{ipx}a^\dagger(\vec{p},\lambda)\right] \qquad E_p = |\vec{p}|$$

We know that in general, in the massless case:

$$\begin{cases} u^\mu(\vec{p}_\Lambda,\lambda)\,e^{-i\theta(\Lambda,p)\lambda} = [D(\Lambda)]^\mu{}_\nu\, u^\nu(\vec{p},\lambda) \\ v^\mu(\vec{p}_\Lambda,\lambda)\,e^{+i\theta(\Lambda,p)\lambda} = [D(\Lambda)]^\mu{}_\nu\, v^\nu(\vec{p},\lambda) \end{cases}$$

We consider $p^\mu = p^\mu_{ref}$ and $\Lambda \equiv \Lambda_\omega \in SE(2)$ $\longrightarrow$ as a consequence $p_\Lambda = \Lambda_\omega p_{ref} = p_{ref}$

$$\begin{cases} u^\mu(\vec{p}_{ref},\lambda)\,e^{-i\theta(\Lambda_\omega,p)\lambda} = [D(\Lambda_\omega)]^\mu{}_\nu\, u^\nu(\vec{p}_{ref},\lambda) \\ v^\mu(\vec{p}_{ref},\lambda)\,e^{+i\theta(\Lambda_\omega,p)\lambda} = [D(\Lambda_\omega)]^\mu{}_\nu\, v^\nu(\vec{p}_{ref},\lambda) \end{cases}$$

Since It seems plausible to assume that A^μ transforms as a 4-vector

$$A^\mu(x) \xrightarrow{\Lambda} \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

as a consequence:

$$\longrightarrow \begin{cases} u^\mu(\vec{P}_{ref},\lambda)\, e^{-i\theta(\Lambda_\omega,P)\lambda} = \Lambda_\omega{}^\mu{}_\nu\, u^\nu(\vec{P}_{ref},\lambda) \\ v^\mu(\vec{P}_{ref},\lambda)\, e^{+i\theta(\Lambda_\omega,P)\lambda} = \Lambda_\omega{}^\mu{}_\nu\, v^\nu(\vec{P}_{ref},\lambda) \end{cases}$$

We see that $v^\mu = (u^\mu)^*$ resolves the system. Therefore:

$$v^\mu = (u^\mu)^* \equiv \varepsilon^\mu(\vec{P}_{ref},\lambda)$$

$$\longrightarrow \boxed{\varepsilon^\mu(\vec{P}_{ref},\lambda)\, e^{-i\theta(\Lambda_\omega,P)\lambda} = \Lambda_\omega{}^\mu{}_\nu\, \varepsilon^\nu(\vec{P}_{ref},\lambda) \qquad \Lambda_\omega \in SE(2)}$$

- To solve this equation we choose $\Lambda_\omega = \Lambda_R(\hat{z},\phi) \longrightarrow \theta(\Lambda_R(\hat{n},\phi),P) = \phi$. It means that:

$$\varepsilon^\mu(\vec{P}_{ref},\lambda)\, e^{-i\phi\lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\mu{}_\nu \varepsilon^\nu(\vec{P}_{ref},\lambda) \qquad \text{with } \vec{P}_{ref} = \kappa\hat{z}$$

We have 4 equations.

$\mu = 0, 3$

$$\begin{cases} \varepsilon^0(\kappa\hat{z},\lambda)(e^{-i\lambda\phi}-1) = 0 \longrightarrow \\ \varepsilon^3(\kappa\hat{z},\lambda)(e^{-i\lambda\phi}-1) = 0 \longrightarrow \end{cases} \boxed{\begin{matrix} \varepsilon^0(\kappa\hat{z},\lambda) = 0 \\ \varepsilon^3(\kappa\hat{z},\lambda) = 0 \end{matrix}} \qquad \lambda = \pm 1$$

$\mu = 1, 2$ (they transform trough the block 2x2)

$$\begin{pmatrix} \varepsilon^1 \\ \varepsilon^2 \end{pmatrix} e^{-i\lambda\phi} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \varepsilon^1 \\ \varepsilon^2 \end{pmatrix} \longrightarrow \begin{cases} i\lambda\varepsilon^1 = \varepsilon^2 \\ -i\lambda\varepsilon^2 = \varepsilon^1 \end{cases} \longrightarrow \boxed{\varepsilon^2(\kappa\hat{z},\lambda) = i\lambda\,\varepsilon^1(\kappa\hat{z},\lambda) \qquad \lambda = \pm 1}$$

We consider $\varepsilon^1(\kappa\hat{z},\lambda) = -\frac{\lambda}{\sqrt{2}} \longrightarrow \varepsilon^2(\kappa\hat{z},\lambda) = \frac{-i}{\sqrt{2}}$

2 transverse polarizations (circular polarizations)

$$\longrightarrow \boxed{\varepsilon^\mu(\kappa\hat{z},-1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \ ; \quad \varepsilon^\mu(\kappa\hat{z},+1) = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}} \Rightarrow \varepsilon^\mu(\kappa\hat{z},\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -\lambda \\ -i \\ 0 \end{pmatrix}$$

At this point we want to generalize to a generic moving frame:

$$\varepsilon^\mu(\vec{P},\lambda) = H\left(\hat{p}, \frac{|\vec{P}|}{\kappa}\right)^\mu{}_\nu \varepsilon^\nu(\vec{P}_{ref},\lambda) = \Lambda_R(\hat{z}\to\hat{p})\,\Lambda_B(\hat{z},\eta)\,\varepsilon^\nu(\vec{P}_{ref},\lambda) = \Lambda_R(\hat{z}\to\hat{p})^\mu{}_\nu\,\varepsilon^\nu(\vec{P}_{ref},\lambda)$$

N.B. We used the fact that $\varepsilon^\nu(\vec{P}_{ref},\lambda)$ is a pure spatial object with only x and y components and therefore it is unaffected by the boost $\Lambda_B(\hat{z},\eta)$.

$$\longrightarrow \boxed{\varepsilon^\mu(\vec{P},\lambda) = \Lambda_R(\hat{z}\to\hat{P})^\mu{}_\nu\,\varepsilon^\nu(\vec{P}_{ref},\lambda)}$$

$$\begin{cases} \mathcal{E}^0(\vec{p},\lambda) = \Lambda_R{}^0{}_\nu \, \mathcal{E}^\nu(\vec{p}_{ref},\lambda) = \Lambda_R{}^0{}_0 \, \mathcal{E}^0(\vec{p}_{ref},\lambda) = \mathcal{E}^0(\vec{p}_{ref},\lambda) = 0 \rightarrow \boxed{\mathcal{E}^0(\vec{p},\lambda) = 0} \\ \mathcal{E}^i(\vec{p},\lambda) = \Lambda_R{}^i{}_j \, \mathcal{E}^j(\vec{p}_{ref},\lambda) = R(\hat{z}\rightarrow\hat{p})\vec{\mathcal{E}}(\vec{p}_{ref},\lambda) \end{cases}$$

Notice that if we compute the scalar product between the spatial parts of p^μ_{ref} and $\mathcal{E}^\mu$ we have:

$$\vec{p}_{ref}\cdot\vec{\mathcal{E}}(\vec{p}_{ref},\lambda) = 0 \qquad \forall\ \lambda = \pm 1$$

Moreover also in the moving frame, since $p^\mu = H^\mu{}_\nu \, p^\nu_{ref} \rightarrow |\vec{p}| = e^{\eta} R(\hat{z}\rightarrow\hat{p})\vec{p}_{ref}$, we have:

$$\boxed{\vec{p}\cdot\vec{\mathcal{E}}(\vec{p},\lambda) = 0} \qquad \forall\ \lambda = \pm 1$$

So we found that:

$$\boxed{\begin{matrix} \mathcal{E}^0(\vec{p},\lambda) = 0 \\ \vec{p}\cdot\vec{\mathcal{E}}(\vec{p},\lambda) = 0 \end{matrix}}$$

This identity translates in:

$$\boxed{A^0(x) = 0 \quad ; \quad \vec{\nabla}\cdot\vec{A}(x) = 0}$$

The fact that $A^0(x)=0$ in any reference frame is a red flag because it would implies that $A^\mu(x)$ cannot transform as a 4-vector, contrary to the initial assumption.

- Let's now consider the case with $\Lambda_\omega \in T_2$. Therefore: $\Lambda_\omega(\alpha,\beta) = e^{-i\alpha T_x - i\beta T_y}$
  We need to know what is the Wigner phase in this case. We remind that a generic Wigner rotation is:

$$H(\hat{p}_\Lambda, \tfrac{|\vec{p}_\Lambda|}{k})^{-1} \Lambda \, H(\hat{p}, \tfrac{|\vec{p}|}{k}) = e^{-i\alpha T_x - i\beta T_y - i\theta J^3}$$

In our case $\vec{p} = \vec{p}_{ref}$ $\Lambda = \Lambda_\omega \longrightarrow \vec{p}_\Lambda = \vec{p}_{ref} \implies H(\hat{p}, \frac{|\vec{p}|}{k}) = \mathbb{1}_{4\times4}$. Therefore:

$$\Lambda_\omega = e^{-i\alpha T_x - i\beta T_y - i\theta J_3} \longleftrightarrow \boxed{\theta = 0}$$

This implies that:

$$\mathcal{E}^\mu(\vec{p}_{ref},\lambda) = \exp\left(-i\alpha \underset{\downarrow}{T_x} - i\beta \underset{\downarrow}{T_y}\right) \mathcal{E}^\nu(\vec{p}_{ref},\lambda)$$

$T_x = J^1_{4vec} + K^2_{4vec}$ $\quad T_y = J^2_{4vec} - K^1_{4vec}$ $\quad \searrow = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -\lambda \\ -i \\ 0 \end{pmatrix}$

$$\left.\begin{matrix} T_x = J^1 + K^1 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\ \\ T_y = J^2 - K^1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{matrix}\right\} \quad -i\alpha T_x - i\beta T_y = \begin{pmatrix} 0 & -\beta & \alpha & 0 \\ -\beta & 0 & 0 & \beta \\ \alpha & 0 & 0 & -\alpha \\ 0 & -\beta & \alpha & 0 \end{pmatrix} \equiv A$$

We see that:

$$A^2 = \begin{pmatrix} \alpha^2+\beta^2 & 0 & 0 & -\alpha^2-\beta^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha^2+\beta^2 & 0 & 0 & -\alpha^2-\beta^2 \end{pmatrix} \quad ; \quad A^3 = 0$$

$$\longrightarrow \exp\left(-i\alpha T_x - i\beta T_y\right) = \mathbb{1} + A + \frac{A^2}{2} = \begin{pmatrix} 1+\frac{1}{2}(\alpha^2+\beta^2) & -\beta & \alpha & -\frac{1}{2}(\alpha^2+\beta^2) \\ -\beta & 1 & 0 & \beta \\ \alpha & 0 & 1 & -\alpha \\ \frac{\alpha^2+\beta^2}{2} & -\beta & \alpha & 1-\frac{1}{2}(\alpha^2+\beta^2) \end{pmatrix}$$

Therefore :

$$\varepsilon^\mu(\vec{p}_{ref},\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-\lambda\\-i\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}\lambda\beta-i\alpha\\0\\0\\\lambda\beta-i\alpha\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-\lambda\\-i\\0\end{pmatrix} + \frac{\lambda\beta-i\alpha}{\sqrt{2}\,K}\begin{pmatrix}K\\0\\0\\K\end{pmatrix} = \varepsilon^\mu(\vec{p}_{ref},\lambda) + \frac{\lambda\beta-i\alpha}{\sqrt{2}}\, p^\mu_{ref}$$

$$\longrightarrow \boxed{\varepsilon^\mu(\vec{p}_{ref},\lambda) = \varepsilon^\mu(\vec{p}_{ref},\lambda) + \frac{\lambda\beta-i\alpha}{\sqrt{2}}\, p^\mu_{ref}}$$

This equation is inconsistent because $\lambda\beta - i\alpha \neq 0$ *if* $\alpha,\beta\in\mathbb{R}$. *Therefore we conclude* that it is not possible to construct a 4-vector quantum field that describes massless spin-1 particles. The only assumption we made, that maybe it is wrong, is that the field transforms like a 4-vector. We have to check that. First of all let's introduce:

$$C_\lambda = \frac{\lambda\beta-i\alpha}{\sqrt{2}\cdot K} \in \mathbb{C} \quad \Rightarrow \quad \varepsilon^\mu(\vec{p}_{ref},\lambda) = \varepsilon^\mu(\vec{p}_{ref},\lambda) + C_\lambda\, p^\mu_{ref}$$

$$\longrightarrow \quad \varepsilon^\mu(\vec{p}_{ref},\lambda) = \Lambda_\omega(\alpha,\beta)^\mu{}_\nu\, \varepsilon^\nu(\vec{p}_{ref},\lambda) = \varepsilon^\mu(\vec{p}_{ref},\lambda) + C_\lambda\, p^\mu_{ref}$$

So the interpretation is that the polarization vectors shift by $C_\lambda p^\mu_{ref}$, so they transform non trivially, under little group translations. Moreover under these type of transformations the physical states $|0, K\hat{z}, \lambda\rangle$ do not transform since $T_x|0,K\hat{z},\lambda\rangle = T_y|0,K\hat{z},\lambda\rangle = 0$. → We find a mismatch with what we discussed previously ★ i.e. that ε^μ do not share the same transformation properties of the physical states.
Since $\varepsilon^\mu(\vec{p}_{ref},\lambda)$ and $\varepsilon^\mu(\vec{p}_{ref},\lambda) + C_\lambda p^\mu_{ref}$ must describe the same physical state we simply require that they do so : we define the pol. vectors as an equivalent class:

$$\left\{\varepsilon^\mu(\vec{p}_{ref},\lambda) = \varepsilon^\mu(\vec{p}_{ref},\lambda) + C_\lambda\, p^\mu_{ref},\ C_\lambda\in\mathbb{C}\right\}$$

Notice that we are not describing a symmetry of the system i.e. a transformation of states which leaves the physics invariant ; this would require that there is a non-trivial transformation of the states.
We can generalize to generic momenta. We know that:

$$\begin{aligned}\varepsilon^\mu(\vec{p},\lambda) &= H\left(\hat{p},\tfrac{|\vec{p}|}{K}\right)^\mu{}_\nu\, \varepsilon^\nu(\vec{p}_{ref},\lambda) = \\ &= H\left(\hat{p},\tfrac{|\vec{p}|}{K}\right)^\mu{}_\nu\, \varepsilon^\nu(\vec{p}_{ref},\lambda) + H^\mu{}_\nu\, C_\lambda\, p^\nu_{ref} = \\ &= \varepsilon^\mu(\vec{p},\lambda) + C_\lambda\, p^\mu\end{aligned}$$

On the other hand on the space of physical states :

$$|0,\vec{p},\lambda\rangle = \mathcal{U}_H\left(\hat{p},\tfrac{|\vec{p}|}{K}\right)|0,\vec{p}_{ref},\lambda\rangle$$

Consequently we are forced to declare that for a generic momentum all polarization vectors which differ by a complex multiple of p^μ represent the same physical states:

$$\boxed{|0,\vec{p},\lambda\rangle \longleftrightarrow \left\{\varepsilon^\mu(\vec{p},\lambda) = \varepsilon^\mu(\vec{p},\lambda) + C_\lambda\, p^\mu,\ C_\lambda\in\mathbb{C}\right\}}$$

In other words we have a redundancy in the description. When translated in the Lagrangian language this redundancy lies at the origin of Gauge invariance. Physics must be invariant if we use $\varepsilon^\mu(\vec{p},\lambda) + c_\lambda p^\mu$ instead of $\varepsilon^\mu(\vec{p},\lambda)$.
This invariance does not correspond to a true symmetry : a symmetry is a transformation on the physical state that leaves the system's physics invariant. Here, on the contrary the states do not transform at all. Therefore is more accurate to speak of redundancy rather than symmetry.

How does $A^\mu(x)$ transform ? $\quad \mathcal{U}(\Lambda)\, A^\mu(x)\, \mathcal{U}(\Lambda)^{-1} = ?$

We know that:

$$1) \quad u^A(\vec{p}_\Lambda,\lambda)\, e^{-i\theta(\Lambda,p)\lambda} = D(\Lambda)^A{}_B\, u^B(\vec{p},\lambda)$$

$$2) \quad \varepsilon^\mu(\vec{p}_\Lambda,\lambda)\, e^{-i\theta(\Lambda,p)\lambda} = \Lambda^\mu{}_\nu\, \varepsilon^\nu(\vec{p},\lambda)$$

$$\longrightarrow (\Lambda^{-1})^\mu{}_\nu\, \varepsilon^\nu(\vec{p}_\Lambda,\lambda) = e^{i\theta(\Lambda,p)\lambda}\, \varepsilon^\mu(\vec{p},\lambda)$$

$$\longrightarrow (\Lambda^{-1})^\mu{}_\nu\, \varepsilon^\nu(\vec{p}_\Lambda,\lambda) + (\Lambda^{-1})^\mu{}_\nu c_\lambda\, p_\Lambda^\nu = e^{i\theta(\Lambda,p)\lambda}\, \varepsilon^\mu(\vec{p},\lambda)$$

$$\longrightarrow (\Lambda^{-1})^\mu{}_\nu\, \varepsilon^\nu(\vec{p}_\Lambda,\lambda) + c_\lambda\, p^\mu = e^{i\theta(\Lambda,p)\lambda}\, \varepsilon^\mu(\vec{p},\lambda)$$

$$\mathcal{U}(\Lambda)\, A^\mu(x)\, \mathcal{U}(\Lambda)^{-1} = \mathcal{U}(\Lambda)\left[\sum_{\lambda=\pm1}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left(\varepsilon^\mu(\vec{p},\lambda)\, e^{-ipx}\, a(\vec{p},\lambda) + h.c.\right)\right]\mathcal{U}(\Lambda)^{-1} =$$

$$= \sum_{\lambda=\pm1}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left(\varepsilon^\mu(\vec{p},\lambda)\, e^{-ipx}\, \mathcal{U}(\Lambda)\, a(\vec{p},\lambda)\, \mathcal{U}(\Lambda)^{-1} + h.c.\right)$$

$$= \sum_{\lambda=\pm1}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\sqrt{\frac{E_{p_\Lambda}}{E_p}}\left(\varepsilon^\mu(\vec{p},\lambda)\, e^{-ipx}\, e^{i\theta(\Lambda,p)\lambda}\, a(\vec{p}_\Lambda,\lambda) + h.c.\right)$$

$$= \sum_{\lambda=\pm1}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\sqrt{\frac{E_{p_\Lambda}}{E_p}}\left(\left[(\Lambda^{-1})^\mu{}_\nu\, \varepsilon^\nu(\vec{p}_\Lambda,\lambda) + c_\lambda\, p^\mu\right] e^{-ipx}\, a(\vec{p}_\Lambda,\lambda) + h.c.\right)$$

$p \to p' = \Lambda p$

$$= \sum_{\lambda=\pm1}\int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sqrt{\frac{E_{p_\Lambda}}{2}}\left[(\Lambda^{-1})^\mu{}_\nu\, \varepsilon^\nu(\vec{p}_\Lambda,\lambda)\, e^{-i\,p_\Lambda\cdot\Lambda x}\, a(\vec{p}_\Lambda,\lambda) + h.c.\right] + \sum_{\lambda=\pm1}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[c_\lambda\, p^\mu e^{-ipx}\, a(\vec{p}_\Lambda,\lambda) + h.c.\right]$$

$$= (\Lambda^{-1})^\mu{}_\nu\, A^\nu(\Lambda x) + \partial_\mu\Omega(\Lambda,x)$$

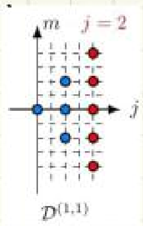
Abelian Gauge transformation

This shift on pol. vect. $\xrightarrow{\text{translates in the fact that}}$ $\boxed{A^\mu(\lambda) \to \mathcal{U}(\Lambda)A^\mu(x)\mathcal{U}(\Lambda)^{-1} = (\Lambda^{-1})^\mu{}_\nu\, A^\nu(\Lambda\lambda) + \partial^\mu\Omega(\Lambda,\lambda}$

$A^\mu(x)$ therefore does not transform as a 4-vector. This transformation is Know as Gauge transformation. Physically it is describing the redundancy in the description.

MASSLESS SPIN-2 FIELD

We need a representation $D(\Lambda)$ such that contains the representation of rotation with $j=2$
A possible guess could be: $D^{(1,1)}$ (the q. field will transform under this representation)
It has 9 d.o.f. As we can see it contains the quintuplet $j=2$ (red dots)
(N.B. We'll be interested in the massless case which corresponds to isolating the states with $\lambda=\pm 2$ from the beforementioned quintuplet.



What is the object which transform under this irreps?

We identified $D^{(\frac{1}{2},\frac{1}{2})}$ with V^μ so a guess would be $D^{(1,1)} \sim T^{\mu\nu}$ where $T^{\mu\nu}$ is a rank-2 tensor: $T'^{\mu\nu}=\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}$. Too large because if we think $T^{\mu\nu}\sim V^\mu V^\nu$ (direct product of vector irr. representations it could be reducible). Let's rewrite it as:

$$T^{\mu\nu} = \underbrace{\frac{1}{2}\left(T^{\mu\nu}+T^{\nu\mu}\right)}_{S^{\mu\nu}} + \underbrace{\frac{1}{2}\left(T^{\mu\nu}-T^{\nu\mu}\right)}_{A^{\mu\nu}}$$

$S^{\mu\nu}$ (10 d.o.f) remains symmetric under Lorentz $S'^{\mu\nu}=\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \hat{S}^{\rho\sigma}$

$A^{\mu\nu}$ (6 d.o.f.) remains anti-symmetric under Lorentz $A'^{\mu\nu}=\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma A^{\rho\sigma}$

$T^{\mu\nu}$ is decomposed in quantities which do not talk to each other. We reduced the number of components from 16 to 10 + 6. Moreover seems that $S^{\mu\nu}$ is the only possible candidate for $D^{(1,1)}$ because $A^{\mu\nu}$ is too small. We need to reduce the d.o.f. of $S^{\mu\nu}$ from 10 to 9.
If we rewrite $S^{\mu\nu}$ as:

$$S^{\mu\nu} = S^{\mu\nu} - \frac{1}{4}g^{\mu\nu}S^\lambda{}_\lambda + \frac{1}{4}g^{\mu\nu}S^\lambda{}_\lambda$$

$$S^{\mu\nu} = \hat{S}^{\mu\nu} + \frac{1}{4}g^{\mu\nu}S^\lambda{}_\lambda$$

we see that $\hat{S}^{\mu\nu}$ is still symmetric and moreover if we compute the trace:

$$g_{\mu\nu}\hat{S}^{\mu\nu} = 0 \quad \text{(traceless) it kills 1 d.o.f.}$$

If we transform $\hat{S}^{\mu\nu}$:

$$\hat{S}'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \hat{S}^{\rho\sigma}$$

we'll get always a symmetric traceless rank-2 tensor. (n.b. trace is scalar → Lorentz invariant)

Therefore we found our guy:

$$\boxed{D^{(1,1)} \sim \hat{S}^{\mu\nu} \equiv h^{\mu\nu}}$$

We can write more explicitily $h^{\mu\nu}$ as:

$$\boxed{h^{\mu\nu}(x) = \sum_{\lambda=\pm 2}\int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[\varepsilon^{\mu\nu}(\vec{p},\lambda)e^{-ip\cdot x}a(\vec{p},\lambda) + h.c.\right]} \qquad E_p=|\vec{p}|$$

Graviton free field

As a consequence the Key equation becomes:

$$\boxed{\mathcal{E}^{\mu\nu}(\vec{p}_{\text{ref}},\lambda)\, e^{-i\theta(\Lambda_w,p)\lambda} = (\Lambda_w)^{\mu}{}_{\rho}(\Lambda_w)^{\nu}{}_{\sigma}\,\mathcal{E}^{\rho\sigma}(\vec{p}_{\text{ref}},\lambda)} \qquad \Lambda_w \in SE(2)\ ;\ \lambda = \pm 2$$

We want to solve this equation. Our guess is that:

$$\boxed{\mathcal{E}^{\mu\nu}(\vec{p}_{\text{ref}},\pm 2) = \mathcal{E}^{\mu}(\vec{p}_{\text{ref}},\pm 1)\cdot\mathcal{E}^{\nu}(\vec{p}_{\text{ref}},\pm 1)} \qquad \text{Gravity} \sim \text{Gauge}^2$$

with $\mathcal{E}^{\mu}(\vec{p}_{\text{ref}},\pm 1)$ which solves the equation for the photon. Using the explicit form of $\mathcal{E}^{\mu}(\vec{p}_{\text{ref}},\pm 1)$ we find that:

$$\boxed{\mathcal{E}^{\mu\nu}(\vec{p}_{\text{ref}},\pm 2) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

as expected it is symmetric and traceless

We can try to generalize it to a generic momentum applying $H^{\mu}{}_{\rho}, H^{\nu}{}_{\sigma}$:

$$\mathcal{E}^{\mu\nu}(\vec{p},\pm 2) = H\left(\hat{p},\tfrac{|\vec{p}|}{\kappa}\right)^{\mu}{}_{\rho}\, H\left(\hat{p},\tfrac{|\vec{p}|}{\kappa}\right)^{\nu}{}_{\sigma}\,\mathcal{E}^{\rho\sigma}(\vec{p}_{\text{ref}},\pm 2)$$

$$\longrightarrow \quad \mathcal{E}^{\mu\nu}(\vec{p},\pm 2) = \left[H\left(\hat{p},\tfrac{|\vec{p}|}{\kappa}\right)^{\mu}{}_{\rho}\,\mathcal{E}^{\rho}(\vec{p}_{\text{ref}},\pm 1)\right]\left[H\left(\hat{p},\tfrac{|\vec{p}|}{\kappa}\right)^{\nu}{}_{\sigma}\,\mathcal{E}^{\sigma}(\vec{p}_{\text{ref}},\pm 1)\right] = \mathcal{E}^{\mu}(\vec{p},\pm 1)\,\mathcal{E}^{\nu}(\vec{p},\pm 1)$$

$$\longrightarrow \quad \boxed{\mathcal{E}^{\mu\nu}(\vec{p},\pm 2) = \mathcal{E}^{\mu}(\vec{p},\pm 1)\,\mathcal{E}^{\nu}(\vec{p},\pm 1)}$$

The field $h^{\mu\nu}$ with the just constructed polarization tensors <u>does not transform covariantly under Lorentz transformations</u>. We indeed have:

$$h^{\mu 0}(x) = 0 \qquad \partial_i h^{ij}(x) = 0$$ they reduce the d.o.f from 3 to 2

which are 2 non-Lorentz invariant constraints. The origin of the problem lies in the transformation properties of the E_2 subgroup of translation of the Little group.
If we take a pure 2-d translation

$$\Lambda_w \equiv \Lambda_w(\alpha,\beta) = e^{-i\alpha T_x - i\beta T_y} \qquad \text{n.b. } \theta(\Lambda_w(\alpha,\beta),p) = 0$$

$$\Lambda_w(\alpha,\beta)^{\mu}{}_{\rho}\,\Lambda_w(\alpha,\beta)^{\nu}{}_{\sigma}\,\mathcal{E}^{\rho\sigma}(\vec{p}_{\text{ref}},\pm 2) = \left[\Lambda_w(\alpha,\beta)^{\mu}{}_{\rho}\,\mathcal{E}^{\rho}(\vec{p}_{\text{ref}},\pm 1)\right]\left[\Lambda_w(\alpha,\beta)^{\nu}{}_{\sigma}\,\mathcal{E}^{\sigma}(\vec{p}_{\text{ref}},\pm 1)\right] =$$

$$= \left[\mathcal{E}^{\mu}(\vec{p}_{\text{ref}},\pm 1) + C_{\pm 1}\, p^{\mu}_{\text{ref}}\right]\left[\mathcal{E}^{\nu}(\vec{p}_{\text{ref}},\pm 1) + C_{\pm 1}\, p^{\nu}_{\text{ref}}\right]$$

$$= \mathcal{E}^{\mu\nu}(\vec{p}_{\text{ref}},\pm 2) + C_{\pm}\, p^{\mu}_{\text{ref}}\,\mathcal{E}^{\nu}(\vec{p}_{\text{ref}},\pm 1) +$$

$$+ \; C_{\pm}\, p^{\nu}_{\text{ref}}\,\mathcal{E}^{\mu}(\vec{p}_{\text{ref}},\pm 1) + C_{\pm}^{2}\, p^{\mu}_{\text{ref}}\, p^{\nu}_{\text{ref}}$$

If we define $v^{\mu}_{\pm}(p_{\text{ref}}) \equiv C_{\pm}\,\mathcal{E}^{\mu}(\vec{p}_{\text{ref}},\pm 1) + \frac{1}{2} C_{\pm}^{2}\, p^{\mu}_{\text{ref}}$, we have:

$$\boxed{\mathcal{E}^{\mu\nu}(\vec{p}_{\text{ref}},\pm 2) = \mathcal{E}^{\mu\nu}(\vec{p}_{\text{ref}},\pm 2) + p^{\mu}_{\text{ref}}\, v^{\nu}_{\pm}(p_{\text{ref}}) + p^{\nu}_{\text{ref}}\, v^{\mu}_{\pm}(p_{\text{ref}})}$$

So we have again that the physical states $|0, k\hat{z}, \pm 2\rangle$ do not transform under Little Group translations but the polarization tensors, which should describe them, do.

It is possible to generalize to a generic momentum:

$$\boxed{\varepsilon^{\mu\nu}(\vec{p},\lambda=\pm 2) = \varepsilon^{\mu\nu}(\vec{p},\lambda=\pm 2) + p^\mu v_\pm^\nu(p) + p^\nu v_\pm^\mu(p) \quad \text{where } v_\pm^\mu(p) \equiv H^\mu{}_\nu v_\pm^\nu(p_{ref})}$$

So we find also in this case a redundancy in the description: the physical states $|0,\vec{p},\lambda=\pm 2\rangle$ are described by polarization tensors not uniquely defined. As in the case of the photon we are again forced to declare that exists an equivalence class of polarization tensors that describe the same physical state

$$\boxed{|0,\vec{p},\lambda=\pm 2\rangle \longleftrightarrow \left\{\varepsilon^{\mu\nu}(\vec{p},\lambda=\pm 2) = \varepsilon^{\mu\nu}(\vec{p},\lambda=\pm 2) + p^\mu v_\pm^\nu(p) + p^\nu v_\pm^\mu(p)\right\}}$$

Considering a generic Lorentz transformation and computing $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \varepsilon^{\rho\sigma}(\vec{p},\pm 2)$ we find that

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \varepsilon^{\rho\sigma}(\vec{p},\lambda) = e^{-i\theta\lambda}\left[\varepsilon^{\mu\nu}(\vec{p}_\Lambda,\lambda) + p_\Lambda^\mu v_\pm^\nu(p_\Lambda) + p_\Lambda^\nu v_\pm^\mu(p_\Lambda)\right] \quad \lambda=\pm 2$$

that we can invert $\longrightarrow$

$$\boxed{(\Lambda^{-1})^\mu{}_\rho (\Lambda^{-1})^\nu{}_\sigma \varepsilon^{\rho\sigma}(p_\Lambda,\lambda) + p^\mu \tilde{v}_\pm^\nu(p) + p^\nu \tilde{v}_\pm^\mu(p) = e^{i\theta\cdot\lambda}\varepsilon^{\mu\nu}(\vec{p},\lambda)}$$

We can use this result to compute $\mathcal{U}(\Lambda) h^{\mu\nu} \mathcal{U}(\Lambda)^{-1}$:

$$\boxed{h^{\mu\nu}(x) \xrightarrow{\Lambda} \mathcal{U}(\Lambda) h^{\mu\nu} \mathcal{U}(\Lambda)^{-1} = (\Lambda^{-1})^\mu{}_\rho (\Lambda^{-1})^\nu{}_\sigma h^{\rho\sigma}(\Lambda x) + \partial^\mu \underbrace{\xi^\nu(x,\Lambda)}_{\text{some function}} + \partial^\nu \underbrace{\xi^\mu(x,\Lambda)}_{\text{some function}}}$$

Gauge transformation for the Graviton field.

this prove that $h^{\mu\nu}$ is not a Lorentz-covariant rank-2 tensor

In conclusion we've learned that if we want to construct a spin-2 massless particles QFT through a field $h^{\mu\nu}$ transforming like a rank-2 symmetric and traceless tensor we must take into account that:

1) It is actually not possible i.e. $h^{\mu\nu}(x)$ will not be a true Lorentz tensor

2) Polarization tensors that differ by a double-shift prop. to the momentum as shown before, must describe the same physics forcing us to takes into account this Gauge redundancy

This fact imposes constraints in the way we construct the interacting theory. Also in the case of tensor field we have restrictions on the kind of coupling that we can have.

If we want to couple the 4vector field V^μ to some current J^μ in order to have a Lorentz invariant object we can do:

$$V^\mu J_\mu$$

If we try to do the same for the photon:

$$A^\mu J_\mu$$

this is <u>not</u> a Lorentz invariant quantity. because $A^\mu J_\mu \xrightarrow{\Lambda} A^\mu J_\mu + (\partial^\mu\Omega)J_\mu$ So we can't couple the photon field with a generi 4-vector current. We could solve the problem finding a way to kill $(\partial^\mu\Omega)J_\mu$. The trick is to couple the photon field with a conserved current i.e. $\partial_\mu J^\mu = 0$. In fact:

$$(\partial^\mu\Omega)J_\mu = \underbrace{\cancelto{0}{\partial^\mu(\Omega J_\mu)}}_{\text{tot derivative}} - \Omega\underset{\overset{\shortparallel}{0}}{\cancel{(\partial_\mu J^\mu)}}$$

The same game is valid for the graviton field : we cannot couple $h^{\mu\nu}$ with a generic rank-2 tensor $T^{\mu\nu}$:

$$h^{\mu\nu}T_{\mu\nu}$$

because this object is not Lorentz scalar because $h^{\mu\nu}$ does not transform as a a true Lorentz tensor (and this is due by the existence of the Gauge redundancy) The trick in this case is to couple $h^{\mu\nu}$ not with a generic rank-2 tensor but with a conserved rank-2 tensor. We'll discover that this tensor is the energy momentum tensor this is why we call $h^{\mu\nu}$ graviton field (because this is what gravity does).

1) We would be tempted to conclude that in QFT every time we want to describe a massless spinning particle we need to introduce Gauge redundancy. Is it true or not?

2) We only discussed consequences of space time symmetries and we classified elementary particles according to spin and mass. However this is not enough : what about charged particles? We need to enlarge our description.

3) If we think about the plot of representations $D^{(j_1,j_2)}$ we only explored repr. which are in the "central strip". What are the other representations? Are they useful for something or not ?

(?) $\mathbb{C}$ repr.

$$\boxed{D^{(0,0)} \quad D^{(1/2,1/2)} \quad D^{(1,1)}}$$

(?) $\mathbb{C}$ repr.

These 3 questions have a very closely related answer.

① Is Gauge redundancy always needed for massless spinning particles? The answer is NO! It is always needed for massless spinning particles in which the representation have $j_1 = j_2$.

<u>Proof</u> (for the exam not all the following computations)

I start from the usual equations:

$$\begin{cases} u^A(\vec{p}_{ref},\lambda)\, e^{-i\theta(\Lambda_w,p)\lambda} = (D(\Lambda_w))^A{}_B\, u^B(\vec{p}_{ref},\lambda) \\ v^A(\vec{p}_{ref},\lambda)\, e^{i\theta(\Lambda_w,p)\lambda} = (D(\Lambda_w))^A{}_B \end{cases} \qquad \Lambda_w \in SE(2)$$

We firstly see for $\Lambda_\omega \in SO(2)$ and then for $\Lambda_\omega \in T_{1,2}$. The point is to prove that the system of equations can be solved only for $j_1 \neq j_2$.

ROTATIONS

$$\begin{array}{l} \Lambda_\omega \equiv \Lambda_\omega(\hat{z},\phi) \\ \theta(\Lambda_\omega, p) = \phi \end{array} \longrightarrow \begin{cases} u^A e^{-i\phi\lambda} = D(\hat{z},\phi)^A{}_B u^B \\ v^A e^{i\phi\lambda} = D(\hat{z},\phi)^A{}_B v^B \end{cases}$$

I Keep the representation generic determined by the pair j_1, j_2

$$D(\hat{z},\phi) \longleftrightarrow D^{(j_1,j_2)}(\hat{z},\phi)$$

an the indeces A and B are pairs of (m_1, m_2). *Therefore I could* rewrithe these eq. as *follows*:

$$\begin{cases} e^{-i\phi\lambda} u^{m_1,m_2} = \underbrace{< m_1\, m_2}_{"A"} | D^{(j_1,j_2)}(\hat{z},\phi) | \underbrace{m_1'\, m_2' >}_{"B"} u^{m_1' m_2'} \\ e^{i\phi\lambda} v^{m_1,m_2} = < m_1, m_2 | D^{(j_1,j_2)}(\hat{z},\phi) | m_1', m_2' > v^{m_1',m_2'} \end{cases}$$

I use the fact that under rot. $D^{(j_1 j_2)} = D^{(j_1)} \otimes D^{(j_2)}$ (basic quantum mechanics); so in both equations I could replace

$$< m_1, m_2 | D^{(j_1,j_2)}(\hat{z},\phi) | m_1', m_2' > = < m_1 | D^{(j_1)}(\hat{z},\phi) | m_1' > < m_2 | D^{(j_2)}(\hat{z},\phi) | m_2' >$$

$$= < m_1 | \exp(-i\phi M^3) | m_1' > < m_2 | \exp(-i\phi N^3) | m_2' >$$

The next step is to consider infinitesimal rotations

- $u^{m_1,m_2}(\cancel{1} - i\phi\lambda) = < m_1 | \mathbb{1} - i\phi M^3 | m_1' > < m_2' | \mathbb{1} - i\phi N^3 | m_2' > u^{m_1' m_2'} =$

$$= (\delta_{m_1 m_1'} - i\phi m_1 \delta_{m_1 m_1'})(\delta_{m_2 m_2'} - i\phi m_2 \delta_{m_2 m_2'}) u^{m_1' m_2'} =$$

$$= \cancel{u^{m_1 m_2}} - i\phi(m_1 + m_2) u^{m_1 m_2} + o(\phi^2) \Longrightarrow \boxed{\lambda = m_1 + m_2} \star$$

- For the other case we get $\boxed{-\lambda = m_1 + m_2} \star$

TRANSLATIONS

$$\begin{array}{l} \Lambda_\omega \equiv \Lambda_\omega(\alpha,\beta) = \exp(-i\alpha T_x - i\beta T_y) \\ \theta(\Lambda_\omega(\alpha,\beta), p) = 0 \end{array} \quad \begin{cases} T_x = J^1 + K^2 \\ T_y = J^2 - K^1 \end{cases} \longrightarrow \begin{cases} u^A(\lambda) = D(\Lambda_\omega(\alpha,\beta))^A{}_B u^B(\lambda) \\ v^A(\lambda) = D(\Lambda_\omega(\alpha,\beta))^A{}_B v^B(\lambda) \end{cases}$$

I would like to rewrite these equations in terms of $\vec{M}$ and $\vec{N}$:

$$\begin{array}{l} \vec{J} = \vec{M} + \vec{N} \\ \vec{K} = i(\vec{N} - \vec{M}) \end{array} \longrightarrow \begin{cases} T_x = (M^1 - iM^2) + (N^1 + iN^2) \\ T_y = (M^2 + iM^1) + (N^2 - iN^1) \end{cases}$$

very similar to ladder operator

- $u^{m_1 m_2} = < m_1 m_2 | D^{(j_1 j_2)}(\alpha,\beta) | m_1' m_2' > u^{m_1' m_2'}$

$$u^{m_1 m_2} = < m_1 m_2 | \exp[-i\alpha(M^1 - iM^2) - i\alpha(N^1 + iN^2) - i\beta(M^2 + iM^1) - i\beta(N^2 - iN^1)] | m_1' m_2' > u^{m_1' m_2'}$$

$$\boxed{\beta=0} \qquad u^{m_1 m_2} = \langle m_1 | \exp\left[-i\alpha\left(M^1 - iM^2\right)\right] | m_1' \rangle \langle m_2 | \exp\left[-i\alpha\left(N^1 + iN^2\right)\right] | m_2' \rangle \, u^{m_1' m_2'}$$

Considering α infinitesimal:

$$u^{m_1 m_2} = \left[\delta_{m_1 m_1'} - i\alpha \langle m_1 | M^1 - iM^2 | m_1' \rangle\right]\left[\delta_{m_2 m_2'} - i\alpha \langle m_2 | N^1 + iN^2 | m_2' \rangle\right] u^{m_1' m_2'}$$

This can be true iff:

$$\longrightarrow \left[\left(M^1 - iM^2\right)_{m_1 m_1'} \delta_{m_2 m_2'} + \delta_{m_1 m_1'} \left(N^1 + iN^2\right)_{m_2 m_2'}\right] u^{m_1' m_2'} = 0$$

$$\longrightarrow \boxed{\left(M^1 - iM^2\right)_{m_1 m_1'} u^{m_1' m_2} + \left(N^1 + iN^2\right)_{m_2 m_2'} u^{m_1 m_2'} = 0}$$

$\boxed{\alpha=0}$, β infinitesimal: $\longrightarrow \boxed{\left(M^2 + iM^1\right)_{m_1 m_1'} u^{m_1' m_2} + \left(N^2 - iN^1\right)_{m_2 m_2'} u^{m_1 m_2'} = 0}$

If we put an i in front the 2nd relation we can see that the 2 equations are visibly related:

$$\begin{cases} \left(M^1 - iM^2\right)_{m_1 m_1'} u^{m_1' m_2} + \left(N^1 + iN^2\right)_{m_2 m_2'} u^{m_1 m_2'} = 0 \\ i\left(M^2 + iM^1\right)_{m_1 m_1'} u^{m_1' m_2} + i\left(N^2 - iN^1\right)_{m_2 m_2'} u^{m_1 m_2'} = 0 \end{cases}$$

We could solve them imposing:

$$\begin{cases} \left(M^1 - iM^2\right)_{m_1 m_1'} u^{m_1' m_2} = 0 & \longrightarrow \boxed{m_1 = -j_1} \\ \left(N^1 + iN^2\right)_{m_2 m_2'} u^{m_1 m_2'} = 0 & \longrightarrow \boxed{m_2 = +j_2} \end{cases}$$

Therefore from the relations ★ we find:

$$\begin{cases} \lambda = m_1 + m_2 = -j_1 + j_2 \\ \lambda = -m_1 - m_2 = j_1 - j_2 \end{cases}$$

If $j_1 = j_2$ (eg $D^{(1/2, 1/2)}, D^{(1,1)} \dots$) then the only possibility is $\lambda = 0$. Instead if $j_1 \neq j_2$ we could solve the system non-trivially.

Example: $j_1 = 1$, $j_2 = 0$ $\longrightarrow$ $\lambda = -1$, $\lambda = +1$

② CHARGE SYMMETRY (à la Wigner - Weyl)

It's time to move to the other possible symmetry: charge symmetry (realized à la Wigner Wile). Let's suppose that we're dealing with a larger symmetry group:

$$\boxed{ISO^{\uparrow}(1,3) \otimes G}$$

where G is a compact Lie group. The generators Q^A commute with those of the Poincaré group:

$$[Q^A, J^{\mu\nu}] = [Q^A, P^\mu] = 0$$

and in particular we also have:

$$[Q^A, P^2] = [Q^A, W^2] = 0$$

The symmetry represented by G is implemented through the following conceptual points:

- $|0\rangle$ is left invariant by the action of all elements of G. i.e.

$$\boxed{Q^A|0\rangle = 0} \qquad \text{(Symmetry realized à la Wigner-Weyl)}$$

- In order to implement this symmetry we use the Wigner theorem → it need to be described by a unitary/antiunitary operators. We need to construct a new unitary representation of the new group. This will be finite because the group is compact. It is given by:

$$\{|M, \vec{p}, s, \lambda\rangle \otimes |\alpha\rangle\}_{M,S,\{\alpha\}}$$

where $|\alpha\rangle$ define a basis of a unitary irrep of G with dimension n_α fully identified by the eigenvalues of the Casimir operators and of other additional commuting operators in the algebra of G. We denote $\{\alpha\}$ the set of these eigenvalues.

Let's take for example $G = SO(2) \longrightarrow [Q^A, Q^B] = i\,\varepsilon_{ABC}\, Q^C \longrightarrow \{|q, q_z\rangle\}_t$

$$\longrightarrow \{|M, \vec{p}, s, \lambda; q, q_z\rangle\}_{M,S,q}$$

<u>What are the consequences?</u>

1) <u>Degeneracy</u>: the subspace tha I added has a dimension of n_α. This extra dimensions imply that I could have more states with same mass and same spin. (n_α degenerate state)

2) <u>Selection Roles</u>: symmetries imply that transitions between different states are forbidden if the quantum numbers are not conserved.

Let's consider the simple case where $G = U(1)$. The most representative concrete case of this example is that of electric charge (but there exist various types of other U(1) symmetries)

Since the group is abelian all complex irreps have dimension 1 and live on $\mathbb{C}$. Consider any of these irreps. We setup the equations. Let's call the generator of G as Q:

$$\boxed{Q|q\rangle = q|q\rangle \quad \text{with } q \in \mathbb{R}} \longrightarrow \boxed{\mathcal{U}(\phi)|q\rangle = e^{-i\phi Q}|q\rangle = e^{-i\phi q}|q\rangle}$$

q represents the charge of the state $|q\rangle$. Imposing that $\mathcal{U}(\phi) = \mathcal{U}(\phi + 2\pi)$ we find that $q \in \mathbb{Z}$. Therefore we could rewrite all as

$$e^{-iQ\phi} = e^{-iQe\frac{\phi}{e}} \equiv e^{-iQe\theta} \quad \text{where } \theta = \frac{\phi}{e} \in [0, \tfrac{2\pi}{e}] \quad \text{and e is the fundamental electric charge } e = 0.13$$

We could define a new generator:

$$\boxed{\tilde{Q} \equiv Q \cdot e} \longrightarrow \boxed{\mathcal{U}(\theta) = e^{-i\tilde{Q}\theta}}$$

$$\longrightarrow \quad \hat{Q}|\tilde{q}\rangle = \tilde{q}|\tilde{q}\rangle \quad \Rightarrow \quad e^{-i\tilde{Q}\theta}|q\rangle = e^{-i\tilde{q}\theta}|\tilde{q}\rangle \longrightarrow \boxed{\tilde{q} = \frac{q}{e} \in \mathbb{Z}}$$

But wait, we know that there exist also fractionary charges like $q_u = \frac{2}{3}e$ or $q_d = -\frac{1}{3}e$. *This fact can be solved allowing projective representations. To do so we should consider the universal covering of* $U(1)$ $: \mathbb{R}$ *; this would imply a continue value of electric charge* ⟶ *this gives the so called problem of quantization of the electric charge* (GUT theories, "Grand Unified theories" *try to solve this*).

We indicate with $|n, \vec{p}, \lambda\rangle$ the states that belong to a given irrep of Poincaré group with n that represents the set of Casimir eigenvalues that define it. Consequently we add the additional label given by the charge q.

$$|n, \vec{p}, \lambda\rangle \otimes |q\rangle \equiv |n, q, \vec{p}, \lambda\rangle$$

Annihilation and creation operators inherit the same q-dependence:

$$|n, q, \vec{p}, \lambda\rangle \equiv \sqrt{2E_p}\, a^+(n, q, \vec{p}, \lambda)|0\rangle \quad ; \quad a(n, q, \vec{p}, \lambda)|0\rangle = 0$$

with

$$\left[a(n, q, \vec{p}, \lambda),\, a^+(n, q, \vec{p}', \lambda')\right]_{\pm} = (2\pi)^3\, \delta(\vec{p} - \vec{p}')\, \delta_{\lambda\lambda'}$$

Moreover the transformation properties of "a" and "a⁺" under the extra $U(1)$ symmetry are:

$$\mathcal{U}(\theta)\, a^+(n, q, \vec{p}, \lambda)\, \mathcal{U}(\theta)^{-1} = e^{-i\theta q}\, a^+(n, q, \vec{p}, \lambda)$$

$$\mathcal{U}(\theta)\, a^+(n, q, \vec{p}, \lambda)\, \mathcal{U}(\theta)^{-1} = e^{+i\theta q}\, a(n, q, \vec{p}, \lambda)$$

Using the infinitesimal form of $\mathcal{U}(\theta) = e^{-i\theta\tilde{Q}}$:

$$\bigstar \begin{cases} \left[Q, a^+(n, q, \vec{p}, \lambda)\right] = +\frac{q}{e}\, a^+(n, q, \vec{p}, \lambda) \\ \left[Q, a(n, q, \vec{p}, \lambda)\right] = -\frac{q}{e}\, a(n, q, \vec{p}, \lambda) \end{cases}$$

We would like to introduce a quantum field that describe particles of species n and charge q.

$$\begin{cases} \mathcal{U}(\Lambda, b)\, \Phi^A(x)\, \mathcal{U}^+(\Lambda, b) = D(\Lambda^{-1})^A{}_B\, \Phi^B(\Lambda x + b) \\ \Phi^A(x) = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left[u^A(n, q, \vec{p}, \sigma)\, e^{-ip\cdot x}\, a(n, q, \vec{p}, \sigma) + v^A(n, q, \vec{p}, \lambda)\, e^{ip\cdot x}\, a^+(n, q, \vec{p}, \lambda)\right] \end{cases}$$

Of course we need to specify also the transformation property of $\Phi^A(x)$ under $\mathcal{U}(1)$ symmetry: (compatibly with the transf. property of a and a^+ under $\mathcal{U}(1)$). *It seems plausible to write* (because $\mathcal{U}(1)$ is just phase *redefinition*):

$$\mathcal{U}(\theta)\, \Phi^A(x)\, \mathcal{U}(\theta)^{-1} = e^{iq_\Phi\theta}\, \Phi^A(x) \overset{\text{infinitesimal}}{\Longrightarrow} \left[Q, \Phi^A(x)\right] = -q_\Phi\, \Phi^A(x)$$

However this is not compatible with ★ (*we could check it*). *If we were to separate* $\Phi^A(x)$ into the 2 components:

$$\begin{cases} \Phi_+^A(x) = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\, u^A(n, q, \vec{p}, \sigma)\, e^{-ip\cdot x}\, a(n, q, \vec{p}, \sigma) \\ \Phi_-^A(x) = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\, v^A(n, q, \vec{p}, \sigma)\, e^{+ip\cdot x}\, a^+(n, q, \vec{p}, \sigma) \end{cases}$$

then it would be possible to make the transformations *compatible* by writing:

$$\bigstar \quad \boxed{\left[Q, \Phi_+^A(x)\right] = -\frac{q_{\Phi_+}}{c}\Phi_+^A(x) \qquad \left[Q, \Phi_-^A(x)\right] = -\frac{q_{\Phi_-}}{e}\Phi_-^A(x)} \quad \text{with } q_{\Phi_+} \equiv +q \text{ and } q_{\Phi_-} \equiv -q$$

It would therefore seems impossible to reconcile the language of QFT with the existence of a conserved charge unless we are forced to double the type of charges present.

Suppose that for each particle species n with charge q we have a corresponding particle species $\bar{n}$ with charge $\bar{q} \equiv -q$. We call $\bar{n}$ the antiparticle of n. We write the corresponding creation and annihilation operators as b^{+} and b:

$$|\bar{n}, \bar{q}, \vec{p}, \lambda > \equiv \sqrt{2E_p}\, b^{+}(\bar{n}, \bar{q}, \vec{p}, \lambda)\, |0\rangle \quad ; \quad b(\bar{n}, \bar{q}, \vec{p}, \lambda)\, |0\rangle = 0$$

with:

$$\left[b(\bar{n}, \bar{q}, \vec{p}, \lambda), b^{+}(\bar{n}, \bar{q}, \vec{p}, \lambda) \right]_{\pm} = (2\pi)^3 \delta(\vec{p} - \vec{q})\, \delta_{\lambda\lambda'}$$

We now have additionaly:

$$\left[Q, b^{+}(\bar{n}, \bar{q}, \vec{p}, \lambda) \right] = +\frac{\bar{q}}{e} b^{+}(\bar{n}, \bar{q}, \vec{p}, \lambda) = -\frac{q}{e} b^{+}(\bar{n}, \bar{q}, \vec{p}, \lambda)$$

$$\left[Q, b\,(\bar{n}, \bar{q}, \vec{p}, \lambda) \right] = -\frac{\bar{q}}{e} b\,(\bar{n}, \bar{q}, \vec{p}, \lambda) = +\frac{q}{e} b\,(\bar{n}, \bar{q}, \vec{p}, \lambda)$$

DIRAC FIELD

We know that
$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \begin{pmatrix} \psi_L(x) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi_R(x) \end{pmatrix} \equiv \Psi_L(x) + \Psi_R(x) \qquad \longleftrightarrow \psi \sim D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$$

Writing down it explicitily:

$$\psi(x) = \sum_{\lambda = \pm \frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left[u(\vec{p}, \lambda) e^{-ipx} a(\vec{p}, \lambda) + v(\vec{p}, \lambda) e^{ipx} b^{+}(\vec{p}, \lambda) \right]$$

where:

$$\begin{cases} u(p, \frac{1}{2}) = \sqrt{2E_p} \begin{pmatrix} 0 \\ 0 \\ \cos\frac{\theta}{2} \\ e^{i\varphi} \sin\frac{\theta}{2} \end{pmatrix} & u(p, -\frac{1}{2}) = \sqrt{2E_p} \begin{pmatrix} \sin\frac{\theta}{2} \\ e^{i\varphi} \sin\frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix} \\ v(p, -\frac{1}{2}) = \sqrt{2E_p} \begin{pmatrix} 0 \\ 0 \\ e^{-i\varphi} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} & v(p, \frac{1}{2}) = \sqrt{2E_p} \begin{pmatrix} e^{-i\varphi} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix} \end{cases}$$

$$\Psi_L(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left[u(p, -\tfrac{1}{2}) e^{-ipx} a(\vec{p}, -\tfrac{1}{2}) + v(p, \tfrac{1}{2}) e^{+ipx} b^{+}(p, \tfrac{1}{2}) \right]$$

$$\Psi_R(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left[u(p, \tfrac{1}{2}) e^{-ipx} a(\vec{p}, \tfrac{1}{2}) + v(p, -\tfrac{1}{2}) e^{+ipx} b^{+}(p, -\tfrac{1}{2}) \right]$$

N.B. In general these commutation relations (infinitesimal form of transf. property of ϕ under G) can be computed as follows:

$$\mathcal{U}(\vec{\alpha}) = \exp(-i\alpha_a Q^a) \longrightarrow \mathcal{U}^{+} \phi\, \mathcal{U} = (\mathbb{1} + i\alpha_a Q^a)\phi^A (\mathbb{1} - i\alpha_b Q^b) \longrightarrow \delta_A{}^B + i\alpha_a [Q^a, \phi^A] = \delta^A{}_B - i\alpha_a (T^a)^A{}_B \phi^{I}$$

$$\longrightarrow \left[Q^a, \phi^A(x) \right] = -(T^a)^A{}_B \phi^B(x)$$

Part 2

Summary of the logic

We defined the physical states via their unitary transformations ; schematically:

$$\boxed{|\psi\rangle \longrightarrow |\psi'\rangle = \mathcal{U}|\psi\rangle} \quad \text{where } \mathcal{U} \in ISO^{\uparrow}(1,3)\times G \quad \begin{cases} \mathcal{U}(\Lambda,a) = \exp(i a_\mu P^\mu)\exp(-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}) \\ \mathcal{U}(\vec{\alpha}) = \exp(-i\alpha_a Q^a) \end{cases}$$

We introduced operators acting on states made out of elementary quantum fields $\Phi^A(x)$. As in ordinary Q.M. we can think about the action of $\mathcal{U}$ on the operators:

$$\langle\varphi|O|\psi\rangle \longrightarrow \langle\varphi'|O|\psi'\rangle = \langle\varphi|\mathcal{U}^\dagger O\,\mathcal{U}|\psi\rangle \equiv \langle\varphi|O'|\psi\rangle \longrightarrow \boxed{O \rightarrow O' \equiv \mathcal{U}^\dagger O\,\mathcal{U}}$$

We postulated the transformation properties of the fields as follows (properties that can be considered more general both in free and interacting theories):

Under Lorentz + spacetime translations

Explicit form:

$$\Phi^A(x) \longrightarrow \Phi'^A(x) = \mathcal{U}^\dagger(\Lambda,a)\Phi^A(x)\mathcal{U}(\Lambda,a) = D(\Lambda)^A{}_B\,\Phi^B(\Lambda^{-1}x - \Lambda^{-1}a) \quad \text{or}$$

Infinitesimal form

$$[P^\mu, \phi^A(x)] = -i\partial^\mu\phi^A(x)$$

$$[J^{\mu\nu}, \phi^A(x)] = -\left[(J_S^{\mu\nu})^A{}_B + J_L^{\mu\nu}\delta^A{}_B\right]\phi^B(x)$$

Under extra symmetries

Explicit form:

$$\Phi^A(x) \longrightarrow \Phi'^A(x) = \mathcal{U}^\dagger(\vec{\alpha})\Phi^A(x)\mathcal{U}(\vec{\alpha}) = \underset{\substack{\uparrow \\ \text{finite dim repr. of } G}}{D(\vec{\alpha})^A{}_B}\Phi^B(\Lambda^{-1}x - \Lambda^{-1}a) \quad \text{or}$$

Infinitesimal form

$$[Q^a, \phi^A(x)] = -(T^a)^A{}_B\,\phi^B(x)$$

In the free theory case the link between $|\psi\rangle \rightarrow |\psi'\rangle = \mathcal{U}|\psi\rangle$ and $O \rightarrow O' = \mathcal{U}^\dagger O\,\mathcal{U}$ was provided by the introduction of creation and annihilation operators

On one hand (multi-)particle states were defined by the action on the vacuum of creation operators

On the other hand Q.fields were defined by a linear superposition of creation and annihilation operators

By requiring consistency between the 2 sides of description we solved the theory: (reconstructing u^A and v^A)

$$\boxed{\Phi^A(x) = \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[u^A(n,\vec{p},\lambda)e^{-ip\cdot x}a(n,\vec{p},\lambda) + v^A(n,\vec{p},\lambda)e^{ip\cdot x}a^\dagger(n,\vec{p},\lambda)\right]}$$

$$H = \sum_n\sum_\lambda\int\frac{d^3\vec{p}}{(2\pi)^3}E_p\,a^\dagger(n,\vec{p},\lambda)a(n,\vec{p},\lambda) \qquad \vec{P} = \sum_n\sum_\lambda\int\frac{d^3\vec{p}}{(2\pi)^3}\vec{p}\,a^\dagger(n,\vec{p},\lambda)a(n,\vec{p},\lambda)$$

We know that these fields satisfy the microcausality condition

$$\boxed{[\Phi^A(x),\Phi^B(y)]_\pm = [\Phi^A(x),\Phi^{\dagger B}(y)]_\pm = 0 \quad (x-y)^2<0}$$

Let's make the following considerations: consider a simple scalar field $\Phi(x)$. We know that:

$$[\Phi(x),\Phi(y)] = 0 \quad \text{if } (x-y)^2<0$$

Consider the equal time commutator: $\left[\Phi(\vec{x},t),\Phi(\vec{y},t)\right]$ for some $t \equiv x^0 = y^0$, since $(x-y)^2 = -|\vec{x}-\vec{y}|^2 < 0$ we have that:

$$\boxed{\left[\Phi(\vec{x},t),\Phi(\vec{y},t)\right] = 0}$$

Furthermore if we consider the free field and its time derivative:

$$\begin{cases} \Phi(\vec{x},t) = \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left(e^{-ipx}a(\vec{p}) + e^{ipx}a^\dagger(\vec{p})\right) \\ \dot{\Phi}(\vec{y},t) = \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[(-iE_p)e^{-ip\cdot x}a(\vec{p}) + (iE_p)e^{ip\cdot x}a^\dagger(\vec{p})\right] \end{cases}$$

and compute:

$$\left[\dot{\Phi}(\vec{x},t),\Phi(\vec{y},t)\right] = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\left[-iE_p\, e^{i(\vec{x}-\vec{y})\cdot\vec{p}} - iE_p\, e^{-i(\vec{x}-\vec{y})\cdot\vec{p}}\right] = -i\,\delta(\vec{x}-\vec{y})$$

$$\Longrightarrow \boxed{\left[\dot{\Phi}(\vec{x},t),\Phi(\vec{y},t)\right] = -i\,\delta(\vec{x}-\vec{y})}$$

Furthermore we can also get:

$$\boxed{\left[\dot{\Phi}(\vec{x},t),\dot{\Phi}(\vec{y},t)\right] = 0}$$

We derived these relations in the free theory. However we notice a suggestive structural similarity with something we know very well. Consider a quantum mechanical system described by a complete set of time dependent dynamical variables $q_a(t)$ and the conjugate momenta $p_a(t)$ which obey universal commutators rules:

$$\boxed{\left[q_a(t),p_b(t)\right] = i\,\delta_{ab} \qquad \left[q_a(t),q_b(t)\right] = \left[p_a(t),p_b(t)\right] = 0}$$

It seems there exists an obvious correspondence:

$$\boxed{\begin{matrix} q_a(t) \xrightarrow{\text{continuum limit}} \Phi(\vec{x},t) \\ p_b(t) \xrightarrow{\qquad\qquad} \dot{\Phi}(\vec{x},t) \end{matrix}}$$

Instead of a discret index a and b we have continuous variables $\vec{x}$ and $\vec{y}$; and as a consequence $\delta_{ab} \longrightarrow \delta(\vec{x}-\vec{y})$ but for the rest the relations are equivalent

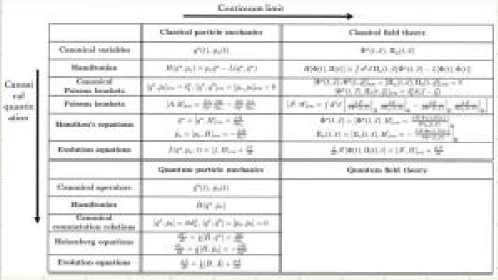
Furthermore if we consider the commutator:

$$\boxed{\left[P^0,\Phi(x)\right] = \left[H,\Phi(x)\right] = -i\partial_t\Phi(x)}$$

It is analogue to the Heisemberg equation of motion in Q.M.

$$\boxed{\frac{dq_a(t)}{dt} = i\left[H,q_a(t)\right]}$$

This analogy allows us to define an Hamiltonian formalism for the fields and "connect" it to the Hamitonian formalism for the states.



Continuum limit

Canonical quantization

	Classical particle mechanics	Classical field theory
Canonical variables	[illegible]	[illegible]
Hamiltonian	[illegible]	[illegible]
Canonical Poisson brackets	[illegible]	[illegible]
Poisson brackets	[illegible]	[illegible]
Hamilton's equations	[illegible]	[illegible]
Evolution equations	[illegible]	[illegible]
	Quantum particle mechanics	Quantum field theory
Canonical operators	[illegible]	
Hamiltonian	[illegible]	
Canonical commutation relations	[illegible]	
Heisenberg equations	[illegible]	
Evolution equations	[illegible]	

How to construct an interacting theory

To construct a generic interacting theory we focus enterily on operators. The idea is to exploit the structural analogy that allows us to define a QFT as the "missing piece" in the box

$$\begin{tikzcd}
\text{CLASSICAL MECHANICS} \arrow[r] \arrow[d] & \text{CLASSICAL FIELD THEORY} \arrow[d] \\
\text{QUANTUM MECHANICS} \arrow[r] & \text{Q.F.T}
\end{tikzcd}$$

- canonical quantization
- continuum limit

Therefore as in the other cases we introduce the formalism of Lagrangians, E.O.M, Hamiltonians. The structural requirement we impose is the following: the theory must have hermitian operators $J^{\mu\nu}, P^{\mu}, Q^{A}$ such that the transformation properties that define the fields are verified

$$\boxed{\begin{aligned} \left[P^{\mu}, \phi^{A}(x)\right] &= -i\partial^{\mu}\phi^{A}(x) \\ \left[J^{\mu\nu}, \phi^{A}(x)\right] &= -\left[(J_S^{\mu\nu})^{A}{}_{B} + J_L^{\mu\nu}\delta^{A}{}_{B}\right]\phi^{B}(x) \\ \left[Q^{a}, \phi^{A}(x)\right] &= -(T^{a})^{A}{}_{B}\phi^{B}(x) \end{aligned}}$$

the possibility to define the operators $P^{\mu}, J^{\mu\nu}, Q^{A}$ is very important also because, if we identify a limit in which the theory becomes free, these operators will generate the unitary transformations that define the multi-particle states.

NÖTHER'S THEOREM

CLASSICAL MECHANICS

Let's consider a system with Lagrangian $L(q^a, \dot{q}^a, t)$, where $q^a(t)$ are the dynamical variables. Let's consider a one-parameter continuous transformation

$$\boxed{q^a(t) \longrightarrow q^a(t,\lambda) \quad ; \; \lambda \in \mathbb{R} \; ; \quad q^a(t, \lambda=0) = q^a(t)}$$

if we consider it infinitesimal:

$$\boxed{q^a(t) \longrightarrow q^a(t, \lambda=0) + \left.\frac{\partial q^a(t,\lambda)}{\partial\lambda}\right|_{\lambda=0}\lambda = q^a(t) + \lambda D q^a(t)}$$

we also know how $\dot{q}^a(t)$ transform:

$$\boxed{\dot{q}^a(t) = \frac{d}{dt}q^a(t) \longrightarrow \dot{q}^a(t) + \lambda\frac{d}{dt}Dq^a(t) = \dot{q}^a(t) + \lambda D\dot{q}^a(t)}$$

The effect on the Lagrangian is:

$$L(q^a + Dq^a\lambda, \dot{q}^a + D\dot{q}^a\lambda, t) - L(q^a, \dot{q}^a, t) = \lambda\left(\frac{\partial L}{\partial q^a}Dq^a + \frac{\partial L}{\partial \dot{q}^a}D\dot{q}^a\right) = \lambda\left(\frac{\partial L}{\partial q^a}Dq^a + P_a\frac{d}{dt}Dq^a\right) \equiv \lambda DL$$

where $$\boxed{DL \equiv \frac{\partial L}{\partial q^a}Dq^a + P_a\frac{d}{dt}Dq^a}$$

Definition: The transformation $q^a(t) \longrightarrow q^a(t,\lambda)$ defines a symmetry i.f.f. we have:

$$\boxed{DL = \frac{dF}{dt}} \quad \text{for some function } F(q^a, \dot{q}^a, t).$$

Comment: it is of course also possible that $F=0$ and in such case $DL=0$
The above condition must hold for arbitrary functions $q^a(t)$ which need not satisfy the E.O.M.

Let's justify the above definition: a symmetry is a transformation of the physical states of a system such that it leaves the physics invariant. In this case the physics is left invariant if the E.O.M. are the same. and the latter are given by the condition that the 1st variational derivative of the action vanishes

$$S[q] = \int dt\ L(q^a(t), \dot{q}^a(t), t) \longrightarrow \boxed{\frac{\delta S[q]}{\delta q^a(t)} = 0} \Longrightarrow \text{E.O.M. describing the dynamics}$$

If the transformation is such that $DL=0$ of course the dynamics is left invariant since if we consider the action:

$$S' = \int_{t_1}^{t_2} dt \left[L(q,\dot{q},t) + \lambda DL \right] = S \quad \text{with } DL=0$$

Therefore if $DL=0$ (the Lagrangian is invariant) → the dynamics is left invariant

There is moreover a 2nd possibility : $DL = \frac{dF}{dt}$. In this case:

$$S' = \int_{t_1}^{t_2} dt\ L(q,\dot{q},t) + \lambda\left(F(t_2) - F(t_1)\right) = S + \lambda\left(F(t_2) - F(t_1)\right)$$

So S and S' differ by a quantity evaluated at the boundary of the integration domain where variations vanishes ⟶ this does not affects to the E.O.M.

Therefore is $DL = \frac{dF}{dt}$ (the Lagrangian is not invariant) but the E.O.M. are the same and the dynamics is left invariant.

Nöther's theorem

Whenever we have such an infinitesimal symmetry there is a conservation law:

$$\boxed{Q = p_a Dq^a - F} \quad \text{does not change the dynamics} \longrightarrow \frac{dQ}{dt} = 0$$

Proof:

$$\frac{dQ}{dt} = \dot{p}_a (Dq^a) + p_a \frac{d}{dt} Dq^a - \frac{dF}{dt} = \qquad \text{E.O.M. } \dot{p}_a = \frac{\partial L}{\partial q^a}$$

$$= \frac{\partial L}{\partial q^a}(Dq^a) + p_a \frac{d}{dt} Dq^a - \frac{dF}{dt} =$$

$$= DL - \frac{dF}{dt} = 0$$

Notice that there's a beautiful "closing of the circle" here:

$$\begin{tikzcd} q^a \longrightarrow q^a + Dq^a \\ DL = \frac{dF}{dt} \arrow[d, bend left=60, "\text{From symmetry to conserved quantity } Q"] \\ Q = p_a Dq^a - F \arrow[u, bend left=60, "\text{From } Q \text{ to } Dq^a"] \end{tikzcd}$$

$$\Rightarrow \boxed{[q^a, Q]_{P.B.} = Dq^a}$$

see the definition! It is completely different from commutators!

FIELD THEORY

Let's sketch now how things work in a field theory. We consider a theory defined by the actio functional:

$$S[\phi] = \int dt\, d^3\vec{x}\, \mathcal{L}(\phi^a(x), \partial_\mu \phi^a(x), x)$$

Our dynamical variables are now the fields $\Phi^a(x)$.

1) We consider a continuous one-parameter set of transformations:

$$\Phi^a(x) \longrightarrow \Phi^a(x,\lambda) \quad ; \lambda \in \mathbb{R} ; \quad \Phi^a(x,\lambda=0) = \Phi^a(x)$$

if we consider it as infinitesimal:

$$\boxed{\Phi^a(x) \longrightarrow \Phi^a(x) + \left.\frac{\partial \Phi^a(x,\lambda)}{\partial \lambda}\right|_{\lambda=0} \lambda = \Phi^a(x) + \lambda D\Phi^a(x)} \qquad D\Phi^a(x) = \frac{\partial \Phi^a(x,\lambda)}{\partial \lambda}$$

the field transformation implies also a transformation in its derivative:

$$\boxed{\partial_\mu \phi^a(x) \longrightarrow \partial_\mu \phi^a(x) + \lambda \partial_\mu (D\phi^a(x))}$$

Therefore the change in the Lagrangian is:

$$\mathcal{L}(\phi^a + \lambda D\phi^a, \partial_\mu \phi^a + \lambda \partial_\mu D\phi^a, x) - \mathcal{L}(\phi^a, \partial_\mu \phi^a, x) = \frac{\partial \mathcal{L}}{\partial \phi^a} \lambda D\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \lambda \partial_\mu D\phi^a \equiv \lambda D\mathcal{L}$$

where $\boxed{D\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^a} D\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\mu D\phi^a}$

Definition: The transformation $\Phi^a(x) \longrightarrow \Phi^a(x,\lambda)$ defines a symmetry i.f.f. we get:

$$\boxed{D\mathcal{L} = \partial_\mu F^\mu} \qquad \text{for some 4-component function } F^\mu = F^\mu(\phi^a, \partial_\mu \phi^a, x)$$

Comment: of course it's possible that $F^\mu = 0$ and in such case we simply have $D\mathcal{L} = 0$

Importantly, the relation above must be valid for arbitrarly field configurations $\phi^a(x)$ which not need to satisfy the E.O.M.

We justify the above definition: a symmetry is a transformation on the physical state of the system such that the physics is left invariant. In this case we seek transformations in the field configuration such that the dynamics of the system, described by the E.O.M. that follow the Hamilton variational principle is left invariant. This of course happens if the transformation is such that $D\mathcal{L} = 0$. However if the transformation is such that $D\mathcal{L} = \partial_\mu F^\mu$ since it is a total derivative it does not affect the action:

$$S \longrightarrow S' = S + \int dt\, d^3\vec{x}\, \partial_\mu F^\mu = S + \int dt\, d^3\vec{x}\, (\partial_t F^0 + \vec{\nabla} \cdot \vec{F})$$

- $\int d^3\vec{x}\, F^0 \Big|_{t_1}^{t_2}$ does not contribute because it is evaluated at the boundary of the domain where variations vanish
- $\int dt\, \vec{F} \cdot \hat{n} \Big|_\infty$ does not contribute because we assume that everything goes to zero sufficiently rapidly in space so that these terms are zero.

We call these transformations infinitesimal global symmetries

Nöther's theorem

For every infinitesimal global symmetry we have a conserved current. More technically it means that it is possible to construct a 4-component object $J^\mu(x)$ such that $\partial_\mu J^\mu = 0$ on the E.O.M.
Parallel to the classical discussion the conserved object will be:

$$\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} D\phi^a - F^\mu}$$

Proof:

$$\partial_\mu J^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \right) D\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\mu D\phi^a - \partial_\mu F^\mu = \quad \text{E.O.M.} \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \right) = \frac{\partial \mathcal{L}}{\partial \phi^a}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi^a} D\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\mu D\phi^a - \partial_\mu F^\mu =$$

$$= D\mathcal{L} - \partial_\mu F^\mu = 0 \qquad \square$$

Discussion

$$\partial_\mu J^\mu = 0 \longrightarrow \partial_t J^0(x) + \partial_i J^i(x) = \partial_t J^0(\vec{x},t) + \vec{\nabla} \cdot \vec{J}(\vec{x},t) = 0$$

If we integrate over some volume V:

$$\partial_t \int_V d^3\vec{x}\, J^0 = -\int_V d^3\vec{x}\, \vec{\nabla} \cdot \vec{J} = -\int_{\partial V} ds\, \vec{J} \cdot \hat{n} \longrightarrow \boxed{\partial_t \int_V d^3\vec{x}\, J^0 = -\int_{\partial V} ds\, \vec{J}(\vec{x},t) \cdot \hat{n}}$$

J^0 : density of "stuff"
$\vec{J}$: current of "stuff"

this equation is telling us that for any volume containing a certain amount of stuff the net change of "stuff" in time depends on the rate at which the "stuff" flows out through the boundaries. This implies that, since "stuff" only leaves one volume to reappear in the adjacent volume, that the total quantity of "stuff" is conserved (assuming that everything goes smoothly to zero at ∞ and therefore we do not have a current at ∞).

We thus find charge conservation

$$\boxed{Q \equiv \int d^3\vec{x}\, J^0(\vec{x},t) \quad ; \quad \frac{dQ}{dt} = 0}$$

Comment: we could rewrite the above result in the following way:

$$J^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^a)} D\phi^a - F^0 = \Pi_a D\phi^a - F^0 \qquad \text{n.b. } \Pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^a)}$$

$$\longrightarrow Q \equiv \int d^3\vec{x}\, (\Pi_a D\phi^a - F^0) = \int d^3\vec{x}\, \Pi_a D\phi^a - F \qquad \text{where } F \equiv \int d^3\vec{x}\, F^0$$

$$\longrightarrow \boxed{Q \equiv \int d^3\vec{x}\, \Pi_a D\phi^a - F}$$

that is formally analogue to the classical result: $Q = P_a Dq^a - F$

Comment : conserved currents are not uniquely defined

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} D\phi^a - F^\mu \quad , \quad D\mathcal{L} = \partial_\mu F^\mu \quad , \quad \partial_\mu J^\mu = 0$$

Suppose indeed we redefine F^μ by adding to it the divergence of some antisymmetric object $A^{\mu\nu}$:

$$F^\mu \longrightarrow F^\mu + \partial_\nu A^{\mu\nu} \quad , \quad A^{\mu\nu} = -A^{\nu\mu}$$

Consequently:

$$\partial_\mu F^\mu \longrightarrow \partial_\mu F^\mu + \partial_\mu \partial_\nu A^{\mu\nu} = \partial_\mu F^\mu$$

so the new F^μ satisfies $D\mathcal{L} = D_\mu F^\mu$ just as well as the old one. However the current J^μ now changed :

$$J^\mu \longrightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} D\phi^a - F^\mu - \partial_\nu A^{\mu\nu} = J^\mu - \partial_\nu A^{\mu\nu} \quad \Rightarrow \boxed{J^\mu \rightarrow J^\mu - \partial_\nu A^{\mu\nu}}$$

we have an alternative definition of the current that is equally valid as the previous one. Importantly, the definition of the total charge Q is unchanged:

$$Q \longrightarrow \int d^3\vec{x}\, (J^0 - \partial_\nu A^{0\nu}) = Q - \cancel{\int d^3\vec{x}\, (\partial_i A^{0i})}^{\,0} = Q$$

Applications and notable results

The idea is the following : remembering the classical circle we push further the analogy ; if we make a field transformation $\phi^a(x) \longrightarrow \phi'^a(x)$ which is a symmetry , we may be able to identify the conserved charges Q with the symmetry generators.

INVARIANCE UNDER SPACE-TIME TRANSLATIONS

$$\boxed{\Phi^A(x) \longrightarrow \Phi'^A(x) = \Phi^A(x - \lambda a)}$$

(it comes from $\phi^A(x) \longrightarrow \phi'^A(x) = D(\Lambda)^A{}_B\, \phi^B(\Lambda^{-1}x - \Lambda^{-1}a)$ with $\Lambda = \mathbb{1}$)

Consider a theory that is described by the action functional:

$$\boxed{S[\phi] = \int dt\, d^3\vec{x}\, \mathcal{L}(\Phi^A(x), \partial_\mu \Phi^A(x))}$$

not explicit dependence on x

Considering λ infinitesimal:

$$\Phi^A(x) \longrightarrow \Phi'^A(x) = \Phi^A(x) - \frac{\partial \Phi^A(x)}{\partial x^\mu} \lambda a^\mu \equiv \Phi^A(x) + \lambda D\Phi^A(x)$$

where the variation of the field is:

$$\boxed{D\Phi^A(x) = -a^\mu \frac{\partial \Phi^A(x)}{\partial x^\mu}} .$$

and :

$$\partial_\mu (D\Phi^A(x)) = -a^\nu \partial_\mu \partial_\nu \Phi^A(x)$$

The change in the Lagrangian is then:

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^A} \partial_\mu \phi^A + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi^A)} \partial_\mu \partial \phi^A$$

$$D\mathcal{L} = -\frac{\partial \mathcal{L}}{\partial \phi^A} a^\mu (\partial_\mu \phi^A) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} a^\mu (\partial_\mu \partial_\nu \phi^A) = -a^\mu (\partial_\mu \mathcal{L}) = -\partial_\mu (g^{\mu\rho} a_\rho \mathcal{L}) \equiv \partial_\mu F^\mu$$

The Lagrangian changes by an amount expressible as $\partial_\mu F^\mu$ with $F^\mu \equiv -g^{\mu\rho} a_\rho \mathcal{L}$
This is therefore an infinitesimal global symmetry.
We now construct the associated conserved current . We apply the definition :

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} D\phi^A - F^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} a^\rho \partial_\rho \phi^A + g^{\mu\rho} a_\rho \mathcal{L} = -a^\rho \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} (\partial_\rho \phi^A) - g^{\mu\rho} \mathcal{L}\right]$$

We made 4 conserved currents depending on the direction in space-time along which we translate. It is then useful to factor out a_ρ and write:

$$J^\mu = -a_\rho T^{\mu\rho}$$

with:

$$\boxed{T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} (\partial^\nu \phi^A) - g^{\mu\nu} \mathcal{L} \quad ; \partial_\mu T^{\mu\nu} = 0}$$

Canonical Energy-Momentum tensor

We have 4 conserved charges obtaining by integrating $T^{0\mu}$ over space-time:

$$\boxed{P^\mu \equiv \int d^3\vec{x}\, T^{0\mu}}$$

Total conserved 4-momentum

- $P^0 = \int d^3\vec{x}\, T^{00} = \int d^3\vec{x} \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^A)} \partial^0 \phi^A - g^{00} \mathcal{L}\right] = \int d^3\vec{x} \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A(x)} \dot{\phi}^A(x) - \mathcal{L}$

After defining the conjugate momentum in the spirit of the canonical formalism

$$\pi_A(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A(\vec{x},t)}$$

we can write:

$$\boxed{P^0 = \int d^3\vec{x} \left(\pi_A(\vec{x},t)\, \dot{\Phi}^A(\vec{x},t) - \mathcal{L}\right) = H}$$

energy

we recognize the Hamiltonian, defined now as the Legendre transform of the Lagrangian that coincides, therefore, with the conserved charge associated to the time translation invariance.

Similarely we find:

$$P^i = \int d^3\vec{x}\, T^{0i} = \int d^3\vec{x} \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^A)} \partial^i \phi^A - \overset{0}{\overset{\shortparallel}{\cancel{g^{0i}}}} \mathcal{L}\right] = \int d^3\vec{x} \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A} (-1) \partial_i \phi^A$$

so that in vector form

$$\vec{P} = -\int d^3\vec{x} \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A} \vec{\nabla} \phi^A$$

and using $\pi_A \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A}$:

$$\boxed{\vec{P} = -\int d^3\vec{x}\, \pi_A(\vec{x},t)\, \vec{\nabla} \phi^A(\vec{x},t)}$$

3-momentum

INVARIANCE UNDER LORENTZ-TRANSFORMATIONS

$$\boxed{\Phi^A(x) \longrightarrow \Phi'^A(x) = \Phi^A(x) + \frac{\omega_{\mu\nu}}{2}\left[-i\,(J_S^{\mu\nu})^A{}_B\,\Phi^B(x) + (x^\mu\partial^\nu - x^\nu\partial^\mu)\,\Phi^A(x)\right]}$$

it comes from
$\Phi^A(x) \longrightarrow \Phi'^A(x) = D(\Lambda)^A{}_B\,\Phi^B(\Lambda^{-1}x)$
by taking Λ infinitesimal

Consider a theory that is described by the action functional:

$$\boxed{S[\Phi] = \int dt\, d^3\vec{x}\, \mathcal{L}(\Phi^A(x), \partial_\mu\Phi^A(x))} \equiv \int dt\, d^3\vec{x}\, \mathcal{L}(x)$$

The variation of the field is given by:

$$\boxed{D\,\Phi^A(x) = \frac{\omega_{\mu\nu}}{2}\left[-i\,(J_S^{\mu\nu})^A{}_B\,\Phi^B(x) + (x^\mu\partial^\nu - x^\nu\partial^\mu)\,\Phi^A(x)\right]}$$

n.b. it originates from the fact that Φ^A has non trivial transformation under Lorentz

n.b. it originates from the space-time dependence x

Consider the simple case in which we have a scalar field. We then have:

$$\boxed{D\Phi(x) = \frac{\omega_{\mu\nu}}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu)\,\Phi(x) = \omega_{\mu\nu}\,x^\mu\,(\partial^\nu\Phi(x))} \quad \bigstar$$

We'll consider theories in which the Lagrangian itself is a scalar object

$$\longrightarrow \quad \boxed{\mathcal{L}(x) \xrightarrow{\Lambda} \mathcal{L}(\Lambda^{-1}x)}$$

this means that the variation of the Lagrangian will take necessarely the form, structorally analogue to $\bigstar$:

$$\boxed{D\mathcal{L} = \omega_{\mu\nu}\,x^\mu\,(\partial^\nu\mathcal{L}(x))}$$

This is a total derivative since we can write:

$$\begin{aligned}
\partial^\nu(\omega_{\mu\nu}x^\mu\mathcal{L}) &= \omega_{\mu\nu}(\partial^\nu x^\mu)\mathcal{L} + \omega_{\mu\nu}x^\mu(\partial^\nu\mathcal{L}) = \\
&= \omega_{\mu\nu}(\partial_\rho g^{\nu\rho}x^\mu)\mathcal{L} + \omega_{\mu\nu}x^\mu(\partial^\nu\mathcal{L}) = \\
&= \omega_{\mu\nu}(g^{\nu\rho}\delta^\mu_\rho)\mathcal{L} + \omega_{\mu\nu}x^\mu(\partial^\nu\mathcal{L}) = \\
&= \cancel{\omega_{\mu\nu}g^{\nu\mu}}\mathcal{L} + \omega_{\mu\nu}x^\mu(\partial^\nu\mathcal{L})
\end{aligned}$$

Consequently we have:

$$D\mathcal{L} = \partial_\mu(g^{\mu\sigma}\omega_{\rho\sigma}x^\rho\mathcal{L}) \equiv \partial_\mu F^\mu \qquad \text{where } F^\mu \equiv \frac{\omega_{\rho\sigma}}{2}(g^{\mu\sigma}x^\rho - g^{\mu\rho}x^\sigma)\mathcal{L}$$

Key point: if we construct the Lagrangian density in such a way that it transforms as a scalar then infinitesimal global Lorentz transformations form a symmetry. *This is not because $\mathcal{L}$ is invariant, $D\mathcal{L} = 0$, but because it changes by a total derivative.*

This is in agreement with what discussed before. If we write:

$$S = \int d^4x\, \mathcal{L}(x)$$

the Lagrangian density under Lorentz transforms according to $\mathcal{L}(x) \xrightarrow{\Lambda} \mathcal{L}(\Lambda^{-1}x)$ so it is not invariant however, at the level of the action:

$$S \to \int d^4x\, \mathcal{L}(\Lambda^{-1}x) = \int d^4x' \, |\det \Lambda|\, \mathcal{L}(x') = \int d^4x\, \mathcal{L}(x) = S \qquad \underline{\text{invariant!}}$$

We now construct the currents. By definition we have:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} D\phi^A - F^\mu =$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \frac{\omega_{\rho\sigma}}{2}\left[-i\left(J_s^{\rho\sigma}\right)^A{}_B \phi^B(x) + (x^\rho \partial^\sigma - x^\sigma \partial^\rho)\phi^A(x)\right] - \frac{\omega_{\rho\sigma}}{2}(g^{\mu\sigma}x^\rho - g^{\mu\rho}x^\sigma)\mathcal{L} =$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \frac{\omega_{\rho\sigma}}{2}(-i)\left(J_s^{\rho\sigma}\right)^A{}_B \phi^B(x) + \underbrace{\frac{\omega_{\rho\sigma}}{2}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(x^\rho \partial^\sigma - x^\sigma \partial^\rho)\phi^A(x) - (g^{\mu\sigma}x^\rho - g^{\mu\rho}x^\sigma)\mathcal{L}\right]}_{(*)} =$$

$$(*) = \frac{\omega_{\rho\sigma}}{2}\left\{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(x^\rho \partial^\sigma \phi^A) - g^{\mu\sigma}x^\rho \mathcal{L}\right] - \rho \leftrightarrow \sigma\right\}$$

$$= \frac{\omega_{\rho\sigma}}{2}\left\{x^\rho\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\partial^\sigma \phi^A - g^{\mu\sigma}\mathcal{L}\right] - \rho \leftrightarrow \sigma\right\}$$

$$= \frac{\omega_{\rho\sigma}}{2}\left\{x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}\right\}$$

and we arrive at the beautiful expression:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \frac{\omega_{\rho\sigma}}{2}(-i)\left(J_s^{\rho\sigma}\right)^A{}_B \phi^B(x) + \frac{\omega_{\rho\sigma}}{2}\left(x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}\right)$$

$$= \frac{\omega_{\rho\sigma}}{2}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(-i)\left(J_s^{\rho\sigma}\right)^A{}_B \phi^B(x) + (x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho})\right]$$

We have 6 independent conserved charges depending on the components of $\omega_{\rho\sigma}$. It is therefore convenient to factor out the dependence on $\omega_{\rho\sigma}$:

$$J^\mu = \frac{\omega_{\rho\sigma}}{2}\mathcal{M}^{\mu\rho\sigma}$$

where we defined the Nöther's current:

$$\boxed{\mathcal{M}^{\mu\rho\sigma} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}(-i)\left(J_s^{\rho\sigma}\right)^A{}_B \phi^B(x) + (x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}) \quad ; \quad \partial_\mu \mathcal{M}^{\mu\rho\sigma} = 0}$$

consequently the Nöther's charges are:

$$\boxed{J^{\mu\nu} \equiv \int d^3\vec{x}\, \mathcal{M}^{0\mu\nu} = \int d^3\vec{x}\left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}^A}(-i)\left(J_s^{\mu\nu}\right)^A{}_B \phi^B(x) + (x^\mu T^{0\nu} - x^\nu T^{0\mu})\right] \quad ; \quad \frac{d}{dt}J^{\mu\nu} = 0}$$

A comment about the interpretation : in classical particle mechanics, invariance under rotations leads to conservation of orbital angular momentum. Let's see what happens in a field theory settings. We consider spatial indices $(\mu,\nu) \longrightarrow (i,j)$ and construct

$$J^i = \frac{1}{2}\varepsilon_{ijk}J^{jk} = \int d^3\vec{x}\left[\frac{\partial\mathcal{L}}{\partial\dot{\phi}^A}(-i)\frac{1}{2}\varepsilon_{ijk}(J_S^{jk})^A{}_B\,\Phi^B + \frac{1}{2}\varepsilon_{ijk}\left(x^jT^{0k}-x^kT^{0j}\right)\right] =$$

$$= \int d^3\vec{x}\left[\underbrace{\frac{\partial\mathcal{L}}{\partial\dot{\phi}^A}(-i)(J_S^i)^A{}_B\,\Phi^B}_{\text{spin density}} + \underbrace{\varepsilon_{ijk}x^jT^{0k}}_{\text{spatial density of orbital angular momentum}}\right]$$

spatial density of 3-mom. $p^k \equiv T^{0k}$

spin density
(it arises only if the fields have non-trivial transformation under Lorentz

spatial density of orbital angular momentum

Consistenly we find that $\vec{J} = \vec{S} + \vec{L}$ and therefore $\vec{J}$ consists of a spin and an orbital angular momentum part.

GLOBAL INTERNAL SYMMETRIES

We now consider the transformation

$$\boxed{\Phi^a(x) \longrightarrow \Phi'^a = \Phi^a(x) - i\alpha_A(T^A)^a{}_b\Phi^b(x)}$$

we want to make it an infinitesimal global symmetry for our theory.
Consider a theory with a Lagrangian density constructed in such a way that it does not transform under the action of G :

$$\mathcal{L} \xrightarrow{G} \mathcal{L}$$

In other words we construct $\mathcal{L}$ using fields that, individually, have non-trivial transformation properties under G but that are combined in a way that leaves $\mathcal{L}$ truly invariant.

<u>Example</u>: Let's take an object H such that :

$$H \xrightarrow{G \equiv SU(2)} \exp\left(-i\frac{\alpha_A\sigma^A}{2}\right)H \qquad \text{n.b. } T^a \equiv \frac{\sigma^a}{2}$$

So to construct $\mathcal{L}$, a possibility could be:

$$\mathcal{L} \sim H^+H \quad \text{in fact} \quad H^+H \longrightarrow H^+\mathcal{U}^+\mathcal{U}H = H^+H \qquad \mathcal{U} = \exp\left(\frac{-i\sigma^a\alpha_a}{2}\right)$$

Under this assumption, we clearly a symmetry since by construction

$$\mathcal{D}\mathcal{L} = 0$$

therefore we can apply Nöther's theorem :

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\mathcal{D}\Phi^a = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\alpha_A(-i)(T^A)^a{}_b\Phi^b = \alpha_A\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\times(-i)(T^A)^a{}_b\Phi^b\right]$$

we have $\dim(G)$ conserved currents.
We can use again the notation:

$$J^\mu \equiv \alpha_A J^{\mu A}$$

where we defined:

$$\boxed{J^{\mu A} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}(-i)(T^A)^a{}_b \Phi^b \quad ; \quad \partial_\mu J^{\mu A} = 0 \quad \forall \ A = 1, \dots, \dim(G)}$$

and the conserved charges therefore are:

$$\boxed{Q^A \equiv \int d^3\vec{x}\ J^{0A} = \int d^3\vec{x}\ \Pi_a(\vec{x},t)(-i)(T^A)^a{}_b \Phi^b(\vec{x},t)}$$

Consistency considerations

1) We would like to check the commutators we'll do it in the case of internal global symmetries. In practic
we would like to compute $[Q^A, \Phi^a(x)] = ?$

We need the quantization conditions: we consider equal-time commutation (or anticommutation) relations:

$$\boxed{\begin{aligned} &[\Phi^A(\vec{x},t), \Pi_B(\vec{y},t)]_\pm = i\,\delta^A{}_B(\vec{x}-\vec{y}) \\ &[\Phi^A(\vec{x},t), \Phi^B(\vec{y},t)]_\pm = [\Pi_A(\vec{x},t), \Pi_B(\vec{y},t)] = 0 \end{aligned}}$$

(We are assuming that, for simplicity all field variables $\Phi^A(\vec{x},t)$ admit a non-vanishing $\Pi_A(\vec{x},t)$).
We then compute:

$$[Q^A, \Phi^A(x)] = \int d^3\vec{y}\,[\Pi_b(t,\vec{y})(-i)(T^A)^b{}_c \Phi^c(\vec{y},t), \Phi^a(\vec{x},t)] =$$

We can use that: $[AB, C] = A[B,C] + [A,C]B = A\{B,C\} - \{A,C\}B$

$$= \int d^3\vec{y}\,[\Pi_b, \Phi^A](-i)(T^A)^b{}_c \Phi^c(\vec{y},t) =$$

$$= \int d^3\vec{y}\cdot i\,\delta^a{}_b\,\delta(\vec{x}-\vec{y})(-i)(T^A)^b{}_c \Phi^c(\vec{y},t) =$$

$$= -(T^A)^a{}_c \Phi^c(\vec{x},t)$$

so we find, as expected:

$$\boxed{[Q^A, \Phi^a(x)] = -(T^A)^a{}_c \Phi^c(x)}$$

2) As a further check, we mentioned that Q_A must be generators of the symmetry group
G. Consequently they must close the algebra:

$$[Q^A, Q^B] = i f_{ABC} Q^C$$

precisely as the T^A do.

In order to do this I need an intermidiate result: (that we can check as exercise):

$$\boxed{[Q^A, \Pi_a(\vec{x},t)] = (T^A)^b{}_a \Pi_b(x)}$$

$$\longrightarrow \left[Q^A, Q^B\right] = \left[Q^A, \int d^3\vec{x}\, \Pi_a(\vec{x},t)(-i)(T^B)^a{}_b\, \Phi^b(\vec{x},t)\right] =$$

$$= \int d^3\vec{x} \left\{ \left[Q^A, \Pi_a\right](-i)(T^B)^a{}_b \Phi^b + (-i)\Pi_a (T^B)^a{}_b \left[Q^A, \Phi^b\right] \right\}$$

$$= \int d^3\vec{x} \left\{ (T^A)^c{}_a \Pi_c \cdot (-i)(T^B)^a{}_b \Phi^b + (-i)\Pi_a (T^B)^a{}_b (-1)(T^A)^b{}_c \Phi^c \right\} =$$

$$= \int d^3\vec{x}\, (-i)(T^A T^B - T^B T^A)^a{}_c\, \Pi_a \Phi^c =$$

$$= i f_{ABC} \int d^3\vec{x}\, \Pi_a (-i)(T^C)^a{}_b \Phi^b =$$

$$= i f_{ABC}\, Q^C \qquad \square$$

So Nöther's charges correspond to generators of the symmetry. N.B. We get this result in the case of the internal global symmetry but it is possible to check that the same relations are valid for the commutators of $J^{\mu\nu}$ and P^μ that close the Poincaré algebra

Interacting Lagrangian for a massive spin-1 field

Now we can try to construct an interacting Lagrangian. The role are the following

- $\mathcal{L}$ must be a scalar, without any explicit "x" dependence
- In natural units $[\mathcal{L}] = M^4$ in fact $[S] = M^0 \longrightarrow$ since $S = \int d^4x\, \mathcal{L}$ and $[L] = M^{-1} \rightarrow [L] = M^4$.
- We construct $\mathcal{L} = \mathcal{L}(\Phi^A, \partial_\mu \Phi^A)$ as an expansion of fields and derivatives. The 1st order in the expansion contains terms quadratic in the fields and terms at most quadratic in the derivative of the fields. This is know as the free theory Lagrangian.
- Important requirement for constructing $\mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))$: it has to satisfy the principle of LOCALITY (will be related to relativistic causality). In practice $\phi(x)$ and $\partial_\mu \phi(x)$ must depend by the same x.

Consider the case of the massive spin-1 field $V^\mu(x)$.

From the propagator we see that:

$$\underbrace{\langle 0|T[V^\mu V^\nu]|0\rangle}_{\text{dimension } [V]^2} = \underbrace{\int \frac{d^4p}{(2\pi)^4} e^{-i\vec{p}(\vec{x}-\vec{y})} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}\right) \frac{i}{p^2 - M^2 + i\varepsilon}}_{\text{dimension } [M]^2} \longrightarrow [V^\mu] = M$$

We apply the procedure and we find:

$$\boxed{\mathcal{L} = -\frac{\alpha}{2}(\partial_\mu V_\nu)(\partial^\mu V^\nu) - \frac{\beta}{2}(\partial_\mu V_\nu)(\partial^\nu V^\mu) + \frac{1}{2} M^2 V_\mu V^\mu} \qquad \text{free theory Lagrangian}$$

We can compute the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial V_\nu} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu V_\nu)} \longrightarrow -\partial_\mu\left[\alpha(\partial^\mu V^\nu) + \beta(\partial^\nu V^\mu)\right] = M^2 V^\nu$$

$$\longrightarrow \alpha \square V^\nu + \beta\, \partial_\mu(\partial^\nu V^\mu) + M^2 V^\nu = 0$$

$$\xrightarrow{\text{we take the 4-divergence}} \alpha\,\Box(\partial_\nu V^\nu) + \beta\,\Box(\partial_\mu V^\mu) + M^2(\partial_\nu V^\nu) = 0$$

$$\longrightarrow \quad \underset{\text{if } \beta=-\alpha}{\cancel{(\alpha+\beta)\Box(\partial_\nu V^\nu)}} + M^2(\partial_\nu V^\nu) = 0 \quad \longrightarrow \quad \boxed{\partial_\mu V^\mu = 0}$$

We obtain precisely the same constraint that we found by analyzing the Lorentz-Poincaré connection. So, if we consider $\alpha=-\beta$ we get:

$$\mathcal{L} = -\frac{\alpha}{2}(\partial_\mu V_\nu)(\partial^\mu V^\nu) + \frac{\alpha}{2}(\partial_\mu V_\nu)(\partial^\nu V^\mu) + \frac{1}{2}M^2 V_\mu V^\mu =$$

$$= -\frac{\alpha}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)(\partial^\mu V^\nu - \partial^\nu V^\mu) + \frac{1}{2}M^2 V_\mu V^\mu$$

we can define

$$\boxed{F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu}$$

and, taking $\alpha=1$ we arrive at the so called PROCA-Lagrangian:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 V_\mu V^\mu}$$

The eq. of motion with this specific choice takes the following form:

$$\boxed{\Box V^\mu - \partial^\mu(\partial_\nu V^\nu) + M^2 V^\mu = 0}$$

$$\xrightarrow{\text{or}} \boxed{\partial_\nu F^{\nu\mu} + M^2 V^\mu = 0}$$

$$\xrightarrow{\text{or}} \boxed{\Box V^\mu + M^2 V^\mu = 0 \quad \text{with} \quad \partial_\mu V^\mu = 0}$$

K.G. equation + constraint (we know very well the physical interpretation of it) It makes 1 d.o.f unphysical (non dynamical)

(Non-dynamical in the language of $\mathcal{L}$ formalism means that there is one field variable whose Π is $\emptyset$.

We now want to introduce interactions. We need to add terms to our free Lagrangian. These interaction terms are made by more than 2 fields.
We classify interactions as follows:

Relevant interactions (a.k.a. SUPER-RENORMALIZIBLE)
They are interactions that consist of operators such that, if we only count the powers of fields and derivatives they have mass dim < 4.
For example:

Exotic interaction: $\mathcal{L}_{int} = \overbrace{\frac{\mu}{3!}\phi^3}^{[M]^4}$

$[\mu]=[M]$ $\quad [\phi]=[M]$

Marginal interactions (a.k.a. RENORMALIZIBLE)
In this case mass dim = 4.
For example:

QED interaction: $\mathcal{L}_{int} = \underset{\uparrow\text{dimensionless}}{e}\,\bar\psi A_\mu \gamma^\mu \psi \qquad [A_\mu]=[M]\,;\quad [\psi]=[M]^{3/2}$

Irrelevant interactions (a.k.a. NON RENORMALIZIBLE)

In this case mass dim >4.

For example:

$$\mathcal{L}_{int} = \frac{1}{\Lambda} \overbrace{F_{\mu\nu}\overline{\psi}\sigma^{\mu\nu}\psi}^{[M]^5} \qquad [\Lambda]=[M]$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad [F]=[M]^2$$

Let's go back to the PROCA-Lagrangian and let's see what happens if I add some interaction:

$$\boxed{\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 V_\mu V^\mu + V_\mu J^\mu}$$

Generic interacting Proca Lagrangian

we only require that J^μ is a 4-vector current so that $V_\mu J^\mu$ is a scalar operator.
Typical examples: (if we consider the interaction with a fermion field)

$$J^\mu = \overline{\psi}\gamma^\mu\psi \quad \text{or} \quad J^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$$

EXERCISE:
Write the Hamiltonian H_0 associated to the free field Proca-Lagrangian an the Hamiltonian associated to the generic interacting Proca Lagrangian.

1) First we consider the free theory. Consider the canonical momenta $\Pi_\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 V^\mu)}$

Since $\frac{\partial\mathcal{L}}{\partial(\partial_\mu V_\nu)} = -F^{\mu\nu} \longrightarrow \frac{\partial\mathcal{L}}{\partial(\partial_0 V_\nu)} = -F^{0\nu}$ consequently $\Pi_0(x)=0 \longrightarrow$ the field variable V^0 therefore is not a canonical field variable, while $\Pi_i(x) = -F_{0i} = +F_{i0} = \partial_i V_0 - \partial_0 V_i$. In 3-dimensional notation:

$$\boxed{\vec{\Pi}(x) = -\vec{\nabla}V^0 - \partial_0\vec{V}}$$

The Hamiltonian is given by the Legendre transform:

$$H = \int d^3\vec{x}\left[\Pi_i(x)\dot{V}^i(x) - \mathcal{L}(V^\mu, \partial_\nu V^\mu)\right]$$

we need to find a way to express V^0 and $\dot{V}^i$ in terms of V^i and Π_i

i) Consider the EOM for the auxiliary field $\boxed{\partial_\nu F^{\nu\mu} + M^2 V^\mu = 0 \quad \text{with } \mu=0}$

$$M^2V^0 + \partial_\nu F^{\nu 0} = M^2 V^0 + \partial_i F^{i0} = 0 \qquad \text{(we solve it for } V^0\text{)}$$

$$\longrightarrow V^0 = -\frac{1}{M^2}(\partial_i F^{i0}) = -\frac{1}{M^2}\partial_i\Pi^i \longrightarrow \boxed{V^0 = -\frac{1}{M^2}\vec{\nabla}\cdot\vec{\Pi}}$$

ii) From the def. of conjugate momenta we have: $\Pi_i = \partial_i V_0 - \partial_0 V_i \longrightarrow$ solve for $\dot{V}^i$ and get:

$$\Pi_i = \partial_i V_0 - \dot{V}_i$$

$$\Pi^i = +\partial^i V^0 - \dot{V}^i \longrightarrow \dot{V}^i = -\partial_i V^0 - \Pi^i$$

or in vector form:

$$\dot{\vec{V}} = -\vec{\nabla}V^0 - \vec{\Pi} = -\vec{\nabla}\left(-\frac{1}{M^2}\vec{\nabla}\cdot\vec{\Pi}\right) - \vec{\Pi} \longrightarrow \boxed{\dot{\vec{V}} = -\vec{\Pi} + \frac{1}{M^2}\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi})}$$

We are now ready to write the Hamiltonian:

$$H = \int d^3\vec{x}\, \mathcal{H}$$

$$\mathcal{H} = -\vec{\Pi}\cdot\dot{\vec{V}} - \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 V_\mu V^\mu\right) =$$

$$= -\vec{\Pi}\cdot\vec{V} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}M^2 V_0 V^0 - \frac{1}{2}M^2 V_i V^i =$$

$$= -\vec{\Pi}\cdot\vec{V} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}M^2\cdot\left(\frac{1}{M^4}\right)(\vec{\nabla}\cdot\vec{\Pi})^2 + \frac{1}{2}M^2|\vec{V}|^2 =$$

$$= -\vec{\Pi}\cdot\left(-\vec{\Pi} + \frac{1}{M^2}\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi})\right) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + \frac{1}{2}M^2|\vec{V}|^2$$

- $\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}F_{0i}F^{0i} + \frac{1}{4}F_{ij}F^{ij} = \frac{1}{2}\Pi_i\Pi^i + \frac{1}{4}F_{ij}F^{ij} = -\frac{1}{2}|\vec{\Pi}|^2 + \frac{1}{4}F_{ij}F^{ij}$
- $\frac{1}{4}F_{ij}F^{ij} = \frac{1}{4}(\partial_i V_j - \partial_j V_i)(\partial^i V^j - \partial^j V^i) = \frac{1}{2}(\partial_i V^i)(\partial^i V^j - \partial^j V^i) = \frac{1}{2}(\partial_i V^j)(\partial_i V^j - \partial_j V^i) = \frac{1}{2}|\vec{\nabla}\times\vec{V}|^2$

In the last relation we used that:

$$(\vec{\nabla}\times\vec{V})\cdot(\vec{\nabla}\times\vec{V}) = \varepsilon_{ijk}\partial_j V^k\, \varepsilon_{ilm}\partial_l V^m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(\partial_j V^k)(\partial_l V^m) = (\partial_j V^k)(\partial_j V^k - \partial_k V^j)$$

$$\longrightarrow \boxed{\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}|\Pi|^2 + \frac{1}{2}|\vec{\nabla}\times\vec{V}|^2}$$

Putting all together we get

$$\mathcal{H} = -\vec{\Pi}\cdot\left(-\vec{\Pi} + \frac{1}{M^2}\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi})\right) - \frac{1}{2M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + \frac{1}{2}M^2|\vec{V}|^2 - \frac{1}{2}|\Pi|^2 + \frac{1}{2}|\vec{\nabla}\times\vec{V}|^2$$

$$= +|\vec{\Pi}|^2 - \frac{1}{2}|\vec{\Pi}|^2 \underbrace{- \frac{1}{M^2}\vec{\Pi}\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi})}_{\substack{\text{(in the integration}\\ \text{by parts survives only}\\ \frac{1}{M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + \text{~~total deriv~~}}} - \frac{1}{2M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + \frac{1}{2}|\vec{\nabla}\times\vec{V}|^2 + \frac{1}{2}M^2|\vec{V}|^2$$

$$\longrightarrow \boxed{\mathcal{H} = \frac{1}{2}|\vec{\Pi}|^2 + \frac{1}{2M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + \frac{1}{2}|\vec{\nabla}\times\vec{V}|^2 + \frac{1}{2}M^2|\vec{V}|^2} \quad \text{Free Proca - Hamiltonian}$$

The fields variables that are canonically quantized are V^i and F_{0i}:

$$\boxed{\begin{aligned} &[V^i(\vec{x},t), V^j(\vec{y},t)] = [F_{i0}(\vec{x},t), F_{j0}(\vec{y},t)] = 0 \\ &[F_{i0}(\vec{x},t), V^j(\vec{y},t)] = -i\,\delta^j_i\,\delta(\vec{x}-\vec{y}) \end{aligned}}$$

We now consider the interaction term:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 V_\mu V^\mu + V_\mu J^\mu}$$

we write the E.O.M as:

$$\boxed{\partial_\mu F^{\mu\nu} + M^2 V^\nu + J^\nu = 0}$$

the conjugate momenta are still:

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 V^\mu)} = -F_{0\mu} \qquad \boxed{\begin{cases} \Pi_0 = 0 \\ \Pi_i = -F_{0i} = +F_{i0} \end{cases}} \begin{array}{l} \longrightarrow V^0 \text{ is still an auxiliary field} \\ \longrightarrow V^i \text{ are still dynamical field} \end{array}$$

i) From the E.O.M for V^0 we get

$$\partial_\mu F^{\mu 0} + M^2 V^0 + J^0 = 0$$
$$\partial_i F^{i0} + M^2 V^0 + J^0 = 0$$
$$\partial_i \Pi^i + M^2 V^0 + J^0 = 0 \qquad \longrightarrow \qquad \boxed{V^0 = -\frac{1}{M^2}\left(\vec{\nabla}\cdot\vec{\Pi} + J^0\right)}$$

ii) From the def. of conjugate momenta:

$$\dot{\vec{V}} = -\vec{\nabla}V^0 - \vec{\Pi} = -\vec{\nabla}\left(-\frac{1}{M^2}\vec{\nabla}\cdot\vec{\Pi} - \frac{1}{M^2}J^0\right) - \vec{\Pi} =$$
$$= -\vec{\Pi} + \frac{1}{M^2}\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi}) + \frac{1}{M^2}\vec{\nabla}J^0 \qquad \longrightarrow \qquad \boxed{\dot{\vec{V}} = -\vec{\Pi} + \frac{1}{M^2}\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi}) + \frac{1}{M^2}\vec{\nabla}J^0}$$

Therefore we write $\mathcal{H}$ as follows;

$$\mathcal{H} = \Pi_i \cdot \dot{V}^i - \mathcal{L} = -\vec{\Pi}\cdot\dot{\vec{V}} - \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 V_\mu V^\mu + J^\mu V_\mu\right) =$$
$$= -\vec{\Pi}\cdot\left(-\vec{\Pi} + \frac{1}{M^2}\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi}) + \frac{1}{M^2}\vec{\nabla}\cdot J^0\right) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}M^2(V_0V^0) - \frac{1}{2}M^2(V_iV^i) - V_0J^0 - V_iJ^i =$$
$$= +|\vec{\Pi}|^2 - \frac{1}{M^2}\vec{\Pi}\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi}) - \frac{1}{M^2}\vec{\Pi}\cdot\vec{\nabla}J^0 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}M^2\left(\frac{1}{M^4}\right)(\vec{\nabla}\cdot\vec{\Pi} + J^0)(\vec{\nabla}\cdot\vec{\Pi} + J^0) + \frac{1}{2}M^2|\vec{V}|^2$$
$$+ \frac{1}{M^2}(\vec{\nabla}\cdot\vec{\Pi} + J^0)J^0 + \vec{V}\cdot\vec{J} =$$
$$= +|\vec{\Pi}|^2 + \frac{1}{M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 - \frac{1}{2}|\vec{\Pi}|^2 + \frac{1}{2}|\vec{\nabla}\times\vec{V}|^2 - \frac{1}{2M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + \frac{1}{2}M^2|\vec{V}|^2 - \frac{1}{M^2}\vec{\Pi}\cdot(\vec{\nabla}J^0) +$$
$$-\cancel{\frac{1}{M^2}(\vec{\nabla}\cdot\vec{\Pi})}J^0 - \frac{1}{2M^2}(J^0)^2 + \cancel{\frac{1}{M^2}(\vec{\nabla}\cdot\vec{\Pi})J^0} + \frac{1}{M^2}(J^0)^2 + \vec{V}\cdot\vec{J} =$$
$$= \frac{1}{2}|\vec{\Pi}|^2 + \frac{1}{2M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + \frac{1}{2}|\vec{\nabla}\times\vec{V}|^2 + \frac{1}{2}M^2|\vec{V}|^2 + \frac{1}{2M^2}(J^0)^2 \underbrace{-\frac{1}{M^2}\vec{\Pi}\cdot\vec{\nabla}J^0}_{\substack{\text{integrating by parts} \\ \text{survives } \frac{1}{M^2}J^0(\vec{\nabla}\cdot\vec{\Pi})}} + \vec{V}\cdot\vec{J}$$

So the Hamiltonian splits in 2 terms: $H = H_0 + H_{int}$

$$\boxed{\begin{aligned} H_0 &= \int d^3\vec{x}\ \frac{1}{2}\left[|\vec{\Pi}|^2 + \frac{1}{M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + |\vec{\nabla}\times\vec{V}|^2 + M^2|\vec{V}|^2\right] \\ H_{int} &= \int d^3\vec{x}\left[\frac{1}{2M^2}(J^0)^2 + \frac{1}{M^2}J^0(\vec{\nabla}\cdot\vec{\Pi}) + \vec{V}\cdot\vec{J}\right] \end{aligned}} \begin{array}{l} \longrightarrow \text{Free Hamiltonian} \\ \\ \longrightarrow \text{Interacting Hamiltonian} \end{array}$$

In QFT we often use the description of interacting theories using the so called interaction picture. Recall the following facts:

$$\left.\begin{aligned}
&\text{SHRÖDINGER PICTURE} && q_s(t) = q_s(0) = q_s \\
&\text{HEISEMBERG PICTURE} && q_H(t) = e^{iHt} q_H(0) e^{-iHt} = e^{iHt} q_s(0) e^{-iHt} \\
&\text{INTERACTION PICTURE} && q_I(t) = e^{iH_0t} q_s(t) e^{-iH_0t} = e^{iH_0t} q_s(0) e^{-iH_0t}
\end{aligned}\right\} \text{at } t=0 \text{ (reference time) the 3 pictures are equivalent}$$

To implement the interaction picture we need to analyze 3 steps:

1) The full Hamiltonian is written in terms of "Heisemberg" fields, for example:

$$\mathcal{U}^{\dagger}(a)\Phi^A(x)\,\mathcal{U}(a) = \Phi^A(x-a) \qquad \text{where } \mathcal{U}(a) = \exp(i a_\mu P^\mu)$$

$a \equiv a_0$, $x-a = x^\mu - a_0$ take $a_0 \equiv t \longrightarrow x^\mu - a_0 = (0, \vec{x})$

$$\longrightarrow e^{-itH}\Phi^A(x)e^{itH} = \Phi^A(0,\vec{x}) \longrightarrow \boxed{\Phi^A(\vec{x},t) = e^{itH}\Phi^A(\vec{x},0)e^{-itH}}$$

2) Since H is independent on time, we can evaluate it at time $t=0$ (the reference time at which Heisemberg and interaction picture coincide)

Consider a simple scalar theory with Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

$$\longrightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \longrightarrow \Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \dot{\phi}$$

and Hamiltonian:

$$H = \int d^3\vec{x}\,(\pi\dot{\phi} - \mathcal{L}) = \int d^3\vec{x}\left(\pi^2 - \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\partial_i\phi)(\partial^i\phi) + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right) =$$

$$= \int d^3\vec{x}\left(\frac{1}{2}\Pi^2 + \frac{1}{2}|\vec{\nabla}\phi|^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)$$

We have:

$$\left\{\begin{aligned}
&H_0 = \int d^3\vec{x}\left[\frac{\Pi(\vec{x},t)^2}{2} + \frac{1}{2}\left(\vec{\nabla}\phi(\vec{x},t)\right)^2 + \frac{1}{2}m^2\phi(\vec{x},t)^2\right] \\
&H_{int} = \int d^3\vec{x}\,\frac{1}{4!}\phi(\vec{x},t)^4
\end{aligned}\right\} \text{they are written in terms of Heisemberg fields}$$

We write at $t=0$

$$\boxed{\left\{\begin{aligned}
&H_0 = \int d^3\vec{x}\left[\frac{1}{2}\Pi_I(\vec{x},0)^2 + \frac{1}{2}\left(\vec{\nabla}\Phi_I(\vec{x},0)\right)^2 + \frac{1}{2}m^2\Phi_I(\vec{x},0)^2\right] \\
&H_{int} = \int d^3\vec{x}\,\frac{1}{4!}\Phi_I(\vec{x},0)^4
\end{aligned}\right.}$$

3) We evolve with the free Hamiltonian to get the interaction picture operators.

$$\boxed{H_{int}(t) = e^{iH_0t}H_{int}e^{iH_0t} = \int d^3\vec{x}\,\frac{1}{4!}\Phi_I(\vec{x},t)^4} \qquad \text{where } \Phi_I(\vec{x},t) = e^{iH_0t}\Phi_I(\vec{x},0)e^{-iH_0t}$$

as tail as H_0 is concerned, we get: $H_0(t) = e^{iH_0t}H_0e^{-iH_0t} = H_0e^{iH_0t}e^{-iH_0t} = H_0$. We can therefore write it at generic time t (since it does not evolve).

$$\boxed{H_0 = \int d^3\vec{x}\left[\frac{1}{2}\Pi_I(\vec{x},t)^2 + \frac{1}{2}\left(\vec{\nabla}\Phi_I(\vec{x},t)\right)^2 + \frac{1}{2}m^2\Phi_I(\vec{x},t)^2\right]}$$

The upshot is that we can simply "replace" in the original $H_0 + H_{int}$ the Heisemberg field operators with interaction-picture fields.

We can apply this procedure to any H we have

$$\longrightarrow \quad H_0 = \int d^3\vec{x}\ \frac{1}{2}\left[|\vec{\Pi}| + \frac{1}{M^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + |\vec{\nabla}\times\vec{V}|^2 + M^2|\vec{V}|^2\right]$$

$$H_{int} = \int d^3\vec{x}\left[\frac{1}{2M^2}(J^0)^2 - V_\mu J^\mu\right]$$

In the interaction term, we rewrite:

$$H_{int} = \int d^3\vec{x}\left[\frac{(J^0)^2}{2M^2} + \underbrace{\frac{J^0(\vec{\nabla}\cdot\vec{\Pi})}{M^2}}_{\substack{\text{using the free theory relation}\\ -M^2V^2 = \vec{\nabla}\cdot\vec{\Pi} \ \longrightarrow\ \frac{J^0(-M^2)V^0}{M^2} = -J^0V_0}} + \vec{V}\cdot\vec{J}\right] = \int d^3\vec{x}\left[\frac{(J^0)^2}{2M^2} \underbrace{- J^0V_0 + \vec{V}\cdot\vec{J}}_{\downarrow\ V_\mu J^\mu = V_0J^0 - \vec{V}\cdot\vec{J}}\right] = \int d^3\vec{x}\left[\frac{1}{2M^2}(J^0)^2 - V_\mu J^\mu\right]$$

Therefore:

$$\boxed{H_{int} = \int d^3\vec{x}\ \frac{1}{2M^2}J^0(\vec{x},t)^2 - \int d^3\vec{x}\ V_\mu J^\mu}$$

So we have <u>2 pieces</u>:

$$\boxed{\begin{aligned} \mathcal{H}^{(2)}_{int} &= -V_\mu J^\mu = -\mathcal{L}_{int} \quad \text{(very nice Lorentz interaction)} \\ \mathcal{H}^{(1)}_{int} &= \frac{1}{2M^2}(J^0)^2 \quad \text{(disturbing term: it breaks the Lorentz invariance)} \end{aligned}}$$

To understand this point let's do some comments

The interaction picture is crucially important in the description of scattering processes

1) In a prototipal scattering element we start off with a state describing a bunch of well separated particles. We indicate this state with $|\psi_{in}\rangle$. We assume these particles to be so well separated that they can be considered as free.

2) We fire the particles at each other and they interact in a complicated way described and governed by the full Hamiltonian

3) The particles then recede from each other and end up in a state in which they are again well separated. We indicate this state with $|\psi\rangle_{out}$. Again we assume the final state particles to be so well separated that they can be considered as free.

4) We are interested in the amplitude for starting from $|\psi\rangle_{in}$ and ending up with $|\psi\rangle_{out}$:

$$\boxed{S_{fi} = {}_{out}\langle\phi|\psi\rangle_{in}}$$

i) In interaction picture both states and op. evolve with time

ii) Operators do that according to free part H_0 : $\Phi_I(t,\vec{x}) = e^{-iH_0t}\,\Phi_I(0,\vec{x})\,e^{-iH_0t}$
and states according to H_{int} : $i\frac{d}{dt}|\psi_I(t)\rangle = H_{int}(t)|\psi_I(t)\rangle$

$$\uparrow$$

$$H_{int}(t) = e^{iH_0t}H_{Int}e^{-iH_0t}$$

iii) The time ev. op. is given by:

$$\boxed{\mathcal{U}_I(t,t') = T\exp\left(-i\int_{t'}^{t}dt''H_I(t'')\right)} \longrightarrow \boxed{|\psi_I(t)\rangle = \mathcal{U}_I(t,t')\,|\psi_I(t')\rangle}$$

The application of the interaction picture to the scattering problem we formulated goes as follows.

1) Faraway from int. region, switch off the interaction $H_0 + H_{int} \to H_0$ therefore the Heisemberg picture op. evolve according to H_0 and states do not evolve. So in this limit int pic. = Heis. pic.

$$\longrightarrow \boxed{\begin{matrix}|\psi_{in}\rangle = & |\psi_I(t=-\infty)\rangle \\ |\phi_{out}\rangle & |\psi_I(t=+\infty)\rangle\end{matrix}} \Rightarrow \boxed{S_{fi} = {}_{out}\langle\Phi|\psi\rangle_{in} = \langle\phi(t=\infty)|\,S\,|\psi_I(t=-\infty)\rangle}$$

$$\downarrow$$

<u>S matrix</u> (operator) (defined by this eq, imposing int pic = Heis pic)

The virtue of the int. picture is that we can find also a formal expression for S. The Key is that the ev. pictures in QM are defined to agree at some reference time $(t=0) \to S_{fi} = \langle\Phi_I(t=0)|\psi_I(t=0)\rangle$

$$S_{fi} = {}_{out}\langle\phi|\psi\rangle_{in} \overset{!}{=} \langle\Phi_I(t=0)|\psi_I(t=0)\rangle \quad \text{where} \begin{cases}|\psi_I(t=0)\rangle = \mathcal{U}_I(0,-\infty)|\psi_I(-\infty)\rangle \\ \langle\Phi_I(t=0)| = \langle\Phi_I(t=\infty)|\,\mathcal{U}_I(\infty,0)\end{cases}$$

$$= \langle\Phi_I(\infty)|\mathcal{U}_I(\infty,0)\,\mathcal{U}_I(0,-\infty)|\psi_I(-\infty)\rangle =$$

$$= \langle\Phi_I(\infty)|\mathcal{U}_I(+\infty,-\infty)|\psi_I(-\infty)\rangle$$

$$\longrightarrow \boxed{S_{fi} = \langle\Phi_I(\infty)|\mathcal{U}_I(+\infty,-\infty)|\psi_I(-\infty)\rangle \quad \text{with} \quad \mathcal{U}_I(+\infty,-\infty) \equiv T\left[\exp\left(-i\int_{-\infty}^{+\infty}dt\,H_{int}(t)\right)\right]}$$

(N.B. in this picture $\mathcal{U}_I$ must be a unitary operator. $\longrightarrow$ $H_I(t)$ must be hermitian)

The main character becomes the so called the Dyson operator :

$$\boxed{\mathcal{U}_I(t,t') = \mathbb{1} - iT\int_{t'}^{t}dt_1\,H_{int}(t_1) + \frac{(-i)^2}{2}T\left[\int_{t'}^{t}dt_1\int_{t_1}^{t}dt_2\,H_{int}(t_2)\,H_{int}(t_1)\right] + \dots}$$

We learned that we start from free H

$$H = \int\frac{d^3\vec{p}}{(2\pi)^3}E_p\,a^\dagger(\vec{p})\,a(\vec{p}) \longrightarrow \mathcal{L}$$

we should check that H is the H that we can obtain by first principles. Let's take a scalar theory: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$; $\dot{\Phi} = \Pi$

$$\longrightarrow H = \int d^3\vec{x} \left[|\vec{\Pi}|^2 + |\vec{\nabla}\cdot\Phi|^2 + m^2 \Phi^2 \right] \longrightarrow H_0 = \int \frac{d^3\vec{p}}{(2\pi)^3} E_p \left[a^\dagger(\vec{p}) a(\vec{p}) + \underset{\substack{\uparrow \\ \text{disturbing term}}}{\tfrac{1}{2}(2\pi)^3 \delta(\vec{0})} \right]$$

If we apply H to vacuum:

$$H_0 |0\rangle = \ldots + \int \frac{d^3p}{\cancel{(2\pi)^3}} \frac{E_p}{2} \cdot \cancel{(2\pi)^3} \delta(\vec{0}) \, |0\rangle = \ldots + \infty$$

$$\int d^3\vec{x} \, e^{-i\vec{k}\cdot\vec{x}} = (2\pi)^3 \delta(\vec{k}) \longrightarrow (2\pi)^3 \delta(\vec{0}) = \int d^3\vec{x} \quad \left(\text{integration over the all space} \rightarrow \text{infinity} \right)$$

We could define the energy density of the vacuum $\mathcal{E}_0 = \lim_{V\to\infty} \frac{E_0}{V} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{E_p}{2} = \infty$ but this integral still diverges! This is know as UV-divergence

(This diverges quite fast $\int d^3\vec{p} \; p^3 \sim p^4$)

Interpretation: we need a prescription to eliminate this ∞. The point is the following. The prescription is called NORMAL ORDERING.

$$: H_0 : \equiv N[H_0]$$

$$: H_0 : = \int \frac{d^3\vec{x}}{2} \left[:\pi^2: + :|\vec{\nabla}\phi|^2: + m^2 :\phi^2: \right]$$

Bosonic : $\phi(x) = \underset{\sim a}{\phi_+(x)} + \underset{\sim a^\dagger}{\phi_-(x)}$; suppose we want to compute

$$\Rightarrow \boxed{:\phi(x)\phi(y): \equiv \phi_-(x)\phi_-(y) + \phi_-(x)\phi_+(y) + \phi_+(x)\phi_+(y) + \phi_-(y)\phi_+(x)}$$

All the creation op. appear on the left and annihilation op. on the right.

Fermionic : $\psi(x) = \underset{\sim a}{\psi_+(x)} + \underset{\sim a^+}{\psi_-(x)}$

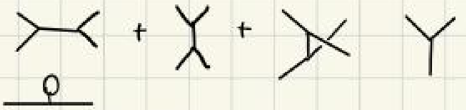
$$\Rightarrow \boxed{:\psi(x)\psi(y): \equiv \psi_-(x)\psi_-(y) + \psi_-(x)\psi_+(y) + \psi_+(x)\psi_+(y) - \psi_-(y)\psi_+(x)}$$

Once we introduce that we should apply everytime we have free field op. If we take the Dyson operator:

$$\mathcal{U}_I(t,t') = \mathbb{1} - i T \int_{t'}^{t} dt_1 : H_{int}(t) : + \frac{(-i)^2}{2!} T \left[\int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 : H_{int}(t_1) : : H_{int}(t_2) : \right] + \ldots$$

If we are only interested in scattering ampl. it's not relevant put the normal ordering (fully conn.) while for a generic element of the scattering matrix we could go through UV divergences if we don't take into account it. (not fully connected interactions)

example of fully conn. int:



example not fully conn int :

Now we want to compute:

$$\underbrace{<\Phi_I(t=\infty)}\,|\,S\,|\underbrace{\psi_I(t=-\infty)>}$$

$$<m_1,\vec{k}_1,\sigma_1;\dots;m_M,\vec{k}_M,\sigma_M|\quad|n_1,\vec{p}_1\lambda;\dots;n_N,\vec{p}_N\,\lambda_N>$$

If the theory is free $S=\mathbb{1}$. If the theory is an int theory:

$$S=\mathbb{1}+i\underset{\uparrow}{T}$$

We are particolarly interested in: T operator

$$\Rightarrow <\Phi_I(t=\infty)|\,iT\,|\psi_I(t=-\infty)> = (2\pi)\,\delta\,(P_{FIN}-P_{IN})\,i\mathcal{M} \quad \text{where} \begin{cases} P_{FIN}=k_1+\dots+k_M \\ P_{IN}=P_1+\dots+P_N \end{cases}$$

invariant matrix element

(this is the quantity computed using Feynman rules at the order which we prefer)

Exercise :

Take the following Hamiltonian $H_{int}=\boxed{\int d^3\vec{x}\,\frac{1}{2M^2}(J^0)^2}-\boxed{\int d^3\vec{x}\,V_\mu J^\mu}$ with $J^\mu=\bar{\psi}\gamma^\mu\psi$.

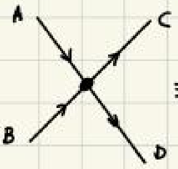
1) Write down the Feynman rules associated to interaction Hamiltonian 1 that comes from ● int. vertex and one from ● int. vertex

$$\downarrow$$

$$\mathcal{H}_{int}=\frac{1}{2M^2}\underbrace{(\bar{\psi}\gamma^0\psi)(\bar{\psi}\gamma^0\psi)}_{6}$$

(4 fermions interaction)

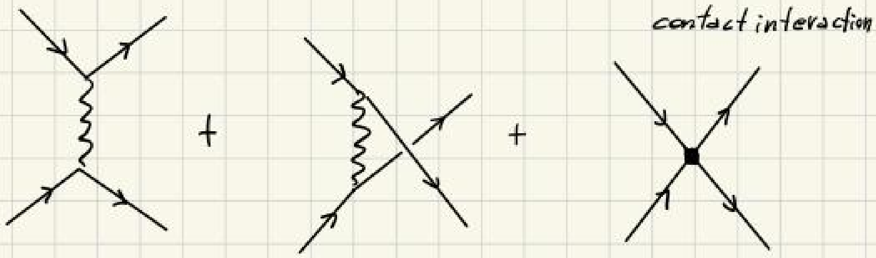
$\equiv ig\gamma^\mu$

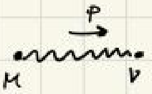


$$\equiv i\frac{g^2}{M^2}\left[(\gamma^0)_{DA}(\gamma^0)_{CB}-(\gamma^0)_{CA}(\gamma^0)_{DB}\right]$$

ABCD are labels for the Dirac components

2) Extract $e^-(p_1,\lambda_1)+e^-(p_2,\lambda_2)\rightarrow e^-(p_3,\lambda_3)+e^-(p_4,\lambda_4)$



N.B. Knowing 

$\equiv\left(-g_{\mu\nu}+\frac{P_\mu P_\nu}{M^2}\right)\frac{i}{P^2-M^2+i\varepsilon}-\frac{i}{M^2}g_{\mu0}g_{\nu0}$ we can show that the 4 fermion interaction part is canceled out by the propagator term.

THE MASSLESS LIMIT OF AN INTERACTING MASSIVE SPIN-1 PARTICLE

Consider again the theory of an interacting massive spin-1 field. We write:

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} M^2 V_\mu V^\mu + V_\mu J^\mu}$$

In principle any current is fine we only require J^μ to be a 4-vector.

We are interested in the limit $M \to 0$. Let's consider the E.O.M. we found:

$$\boxed{\partial_\mu F^{\mu\nu} + M^2 V^\nu + J^\nu = 0}$$

Does this equation of motion have a sensible zero-mass limit?

We take the 4 divergence, and obtain:

$$\cancel{\partial_\nu \partial_\mu F^{\mu\nu}} + M^2 \partial_\nu V^\nu + \partial_\nu J^\nu = 0 \xrightarrow[M\to 0]{\text{in order to make sense the limit for}} \text{we impose } \boxed{\partial_\mu J^\mu = 0}$$

because of antisymmetry of $F^{\mu\nu}$

So we learn a simple fact: in order, for the interacting massive spin-1 theory to make sense the limit $M\to 0$, we need to have a coupling with a conserved current.
A question naturally arises. We mentioned that in the limit $M \to 0$ we have troubles with the longitudinal polarization; what happens to this problem? To investigate this question we can look at the simplest process allowed by the theory we're considering: consider the amplitude for the production of a massive vector of momentum $\vec{k}$ and helicity λ. For simplicity we consider $J^\mu(x)$ to be a classical source i.e. it is not an operator but only a c-number four-vector. A prototypal example could be the source current due to an heavy nucleus at rest in some position in space:

$$J^\mu(x) = \begin{cases} J^0(\vec{x}) = Z \cdot e\, \delta(\vec{x}) \\ \vec{J}(\vec{x}) = \vec{0} \end{cases}$$

Consider the scattering amplitude with states $|in\rangle = |0\rangle$, $|fin\rangle = |V(\vec{k},\lambda)\rangle$ with $E_k = \sqrt{|\vec{k}|^2 + M^2}$.

$$S_{fi} = \langle V(\vec{k},\lambda)| (-i) \int dt\, H_{int}(t) |0\rangle \quad \text{with} \quad H_{int} = \int d^3\vec{x}\, (-1) V_\mu J^\mu$$

$$= \langle V(\vec{k},\lambda)|\, i \int d^4x\, V_\mu J^\mu(x) |0\rangle = (*)$$

But since $V^\mu = V_+^\mu + V_-^\mu$ only the contribution from V_-^μ survives (because it has a^+)

$$V_-^\mu = \sum_{\lambda'} \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} a^+(\vec{p},\lambda') \mathcal{E}^\mu(\vec{p},\lambda')^* e^{ip\cdot x} \longrightarrow \langle V(\vec{k},\lambda)| V_-^\mu = \langle 0| \mathcal{E}^\mu(\vec{k},\lambda)^* e^{ik\cdot x}$$

$$\longrightarrow \quad (*) = i\, \mathcal{E}_\mu^*(\vec{k},\lambda) \int d^4x\, e^{ik\cdot x} \langle 0| J^\mu(x) |0\rangle =$$

$$= i\, \mathcal{E}_\mu^*(\vec{k},\lambda) \int d^4x\, e^{ik\cdot x} J^\mu(x) =$$

$$= i\, \mathcal{E}_\mu^*(\vec{k},\lambda)\, \tilde{J}^\mu(x) \longrightarrow \boxed{S_{fi} = i\, \mathcal{E}_\mu^*(\vec{k},\lambda)\, \tilde{J}^\mu(x)}$$

Let's consider the case in which k^μ is directed along the $\hat{z}$ direction

$$\longrightarrow \boxed{k^\mu = (E_k, 0, 0, |\vec{k}|) \quad ; \quad E_k = \sqrt{|\vec{k}|^2 + M^2}}$$

We also consider specifically the amplitude for producing a longitudinal polarized massive vector:

$$\boxed{\mathcal{E}^\mu(|\vec{k}|\hat{z}, \lambda=0) = \begin{pmatrix} |\vec{k}|/M \\ 0 \\ 0 \\ E_k/M \end{pmatrix}}$$

Knowing that $\partial_\mu J^\mu(x)=0$ this means that in Fourier space : $\boxed{k_\mu \hat{J}(k)=0}$

$$\rightarrow \mathcal{E}^*_\mu(|\vec{k}|\hat{z},\lambda=0)\,\tilde{J}^\mu(k) = \frac{|\vec{k}|}{M} J^0(k) - \mathcal{E}^i(|\vec{k}|\hat{z},\lambda=0)\,\tilde{J}^i(k) = \frac{|\vec{k}|}{M} J^0(k) - \frac{E_k}{M}\tilde{J}^3(k) = \frac{1}{M}\left(|\vec{k}|\tilde{J}^0(k) - \sqrt{|\vec{k}|^2+N^2}\,\tilde{J}^3(k)\right)$$

$$\longrightarrow \boxed{\mathcal{E}^*_\mu(|\vec{k}|\hat{z},\lambda=0)\,\tilde{J}^\mu(k) = \frac{1}{M}\left(|\vec{k}|\tilde{J}^0(k) - \sqrt{|\vec{k}|^2+N^2}\,\hat{J}^3(k)\right)} \;\bigstar$$

We now use the conservation equation that reads:

$$K_0\tilde{J}^0(K) + K_i\tilde{J}^i(K) = K_0\tilde{J}^0(K) - K^i\tilde{J}^i(K) = E_K\tilde{J}^0(K) - |\vec{k}|\tilde{J}^3(K) \overset{!}{=} 0 \longrightarrow \boxed{\tilde{J}^3(K) = \frac{E_K}{|\vec{k}|}\tilde{J}^0(K)}$$

Substituting it into ★ we find:

$$i\,\mathcal{E}^*_\mu(K,\lambda)\,\tilde{J}^\mu(K) = \frac{1}{M}\left(|\vec{k}|\tilde{J}^0 - \frac{\sqrt{|\vec{k}|^2+M^2}}{|\vec{k}|}E_K\tilde{J}^0(K)\right)$$

Therefore:

$$S_{fi} = i\left(|\vec{k}| - \frac{\sqrt{|\vec{k}|^2+M^2}}{|\vec{k}|}E_K\right)\tilde{J}^0(K) = \frac{i}{M}\tilde{J}^0(K)\left[\cancel{|\vec{k}|} - \cancel{|\vec{k}|} - \frac{M^2}{|\vec{k}|}\right] \longrightarrow \boxed{S_{fi} = -i\frac{M}{|\vec{k}|}\tilde{J}^0(K)}$$

We find something extremely interesting. In our discussion about the free theory for a massive spin-1, we found that the formalism collapses if we take the limit $\frac{M}{|\vec{k}|}\to 0$. This is because in this limit the longitudinal polarization blows up. At that time, we said that we cannot take the limit $M\to 0$ and obtain the massless spin-1 from the massive spin-1.

On a pure theoretical side, we also understood well the origin of the obstruction. The origin of the obstruction lies in the fact that massive spin-1 particles have 3 d.o.f. while massless spin-1 only 2, and a description of massless spin-1 particles with a vector field $V^\mu(x)$ is impossible without introducing Gauge redundancy, which is not needed in the massive case.

So far so good from the theory side. However, the above discussions prompts an obvious question. Consider a massless spin-1 and a massive spin-1 but with a super tiny mass say 10^{-23}eV. Since the theories are so different, it should be possible to see a huge difference even if the value $M\neq 0$ is so small.

This is not true. To answer the above doubt we need an interacting theory. We could see that if we compute the amplitude for a physically reasonable, very simple process we find that the contribution of the longitudinal polarization, that is the one that could tell the difference between the massive and massless case - vanishes as $\frac{M}{|\vec{k}|}\to 0$. Consequently we find a physically reasonable answer: a massive vector with mass 10^{-23} eV looks like a photon and, despite the 3 helicity d.o.f., only the transverse one are taken.

We also learnt another important lesson we can define an interacting theory for a massless photon by taking the limit $M\to 0$ from a massive theory coupled to a conserved current

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2V_\mu V^\mu + V_\mu J^\mu \xrightarrow[\text{with } \partial_\mu J^\mu=0]{M\to 0} \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu \;;\; F_{\mu\nu}=\partial_\mu A_\nu - \partial_\nu A_\mu \;;\; \partial_\mu J^\mu = 0}$$

In this limit we just decouple the longitudinal d.o.f. and the free photon field becomes:

$$A^\mu(x) = \sum_{\lambda=\pm}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[\mathcal{E}^\mu(\vec{p},\lambda)e^{-ip\cdot x}a(\vec{p},\lambda) + \text{h.c.}\right]$$

Interacting Photon and Gauge invariance

The need to couple the photon field to a conserved current can also be derived without going through the massless limit of the spin-1 field. As we have discussed, the physical photon field is not a 4-vector, since:

$$\boxed{A^\mu(x) \longrightarrow \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) + \partial^\mu \Omega(x)}$$

Consequently, if we follow our rules for the construction of a $\mathcal{L}$, terms like:

- $\frac{1}{2} M^2 A_\mu A^\mu$ or $A_\mu J^\mu$ are forbidden since not Lorentz invariant

- The absence of the 1st term is very welcome since it explains why the photon has no mass.
- The absence of the 2nd term is more problematic since it forbids interactions.

To solve this problem we have 2 possibilities:

1) We abandon the description in terms of $A^\mu(x)$ and look for a description which does not require Gauge conditions. As we discussed, it is indeed possible to describe massless spin-1 not introducing Gauge redundancy. The semplest option is to consider the representation $(1,0)$ or $(0,1)$. Since we want a parity-invariant theory we consider the direct sum $(1,0) \oplus (0,1)$. Objects that transform in this way are Lorentz antisymmetric rank-2 tensors. This is the case of $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ that can be written explicitily as:

$$F^{\mu\nu} = \sum_{\pm} \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left[u^{\mu\nu}(\vec{p},\lambda)\, e^{-ip\cdot x} a(\vec{p}) + \text{h.c.} \right]$$

$$u^{\mu\nu}(\vec{p},\lambda) = p^\mu \varepsilon^\nu(\vec{p},\lambda) - p^\nu \varepsilon^\mu(\vec{p},\lambda)$$

We could proof that it is actually a true Lorentz tensor. Let's see which relations it satisfies:

- $\boxed{\partial_\mu F^{\mu\nu} = 0}$ in fact by K.G. $\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \overset{\overset{0}{\parallel}\ \text{(massless K.G. eq)}}{\cancel{\Box A^\nu}} - \partial^\nu(\partial_\mu A^\mu) =$
  $= -\partial^\nu(\partial_0 A^0 + \partial_i A^i) = 0$ Since $A^0 = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$

- $\boxed{\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0}$ in fact by antisymmetry $\varepsilon^{\mu\nu\rho\sigma} \partial_\nu(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = 0$

These are nothing but the Maxwell equations in vacuum and they can be derived from the Lagrangian density:

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

There is no need of Gauge redundancy; however the problem are interactions. The only interactions allowed to involve $F_{\mu\nu}$ take the form:

$$F_{\mu\nu} \bar{\psi} \sigma^{\mu\nu} \psi \quad , \quad F_{\mu\nu} \bar{\psi} \gamma^5 \sigma^{\mu\nu} \psi$$

we cannot construct interactions with $\bar{\psi}\gamma^\mu\psi$ and therefore we do not recover classical E.M. (to get Coolomb we need exactly $\bar{\psi}\gamma^\mu\psi$).

2) The other possibility is to insist using $A^\mu(x)$ but we add an additional requirement In order to be allowed, a term must be not only "Lorentz" invariant looking" but also invariant under the transformation

$$\boxed{A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x)} \quad \text{for a generic } \alpha(x)$$

Consequently, the piece that one gets from the Lorentz transformation of $A_\mu(x)$ will be harmless. Schematically:

Lorentz looking invariant
+
invariant under the formal replacement $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$ $\longrightarrow$ Truly Lorentz invariant

Consequently an interaction term of the form $A_\mu J^\mu$ is not allowed since is not invariant under the formal rescaling $A_\mu \longrightarrow A_\mu + \partial_\mu \alpha$ (for a generic J^μ).
However conseder an interaction term of the kind $\mathcal{L}_{int} = A_\mu J^\mu$ with J^μ conserved 4-current. This term is surely "Lorentz-invariant looking" but it is also invariant under the formal replacement $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$, since:

$$A_\mu J^\mu \longrightarrow (A_\mu + \partial_\mu \alpha) J^\mu = A_\mu J^\mu + (\partial_\mu \alpha) J^\mu = A_\mu J^\mu - \cancel{\alpha(x)(\partial_\mu J^\mu)} = A_\mu J^\mu$$

integrating by parts and throwing away the boundery term we are left with $\partial_\mu J^\mu = 0$

Consequently we obtain the same result as before. The interacting Lagrangian takes the form:

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \quad , \quad \partial_\mu J^\mu = 0}$$

The question is now: How do we construct such conserved current? Clearly the obvious way is to start with a theory that enjoys a global symmetry and the after constructing the corresponding current J^μ, couple it with A_μ. This works just fine in QED.
Consider a Dirac field with the Lagrangian

$$\boxed{\mathcal{L} = \bar{\psi}(i\not{\partial} - m_\psi)\psi} \qquad \text{This } \mathcal{L} \text{ enjoys a global } U(1) \text{ symmetry}$$

If we transform

$$\boxed{\psi(x) \rightarrow e^{i\alpha}\psi(x)} \quad \text{with } \alpha \in \mathbb{R}$$

the Lagrangian is invariant. We know very well how to interpret this symmetry physically. This is a Wigner-Weil symmetry which describes the presence of charged fermions. There is actually more. If we want to describe charged particles with charge $q \cdot e$ (an antiparticles with $-qe$), we require that ψ transform according to:

$$\boxed{\psi(x) \rightarrow e^{-iqe\theta}\psi(x)}$$

At the infinitesimal level:

$$\psi(x) \longrightarrow \psi'(x) = \psi(x) - iqe\theta\psi(x) \longrightarrow \boxed{D\psi(x) \equiv -iqe\psi(x)}$$

The Noether current is:

$$J^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} D\psi = (\bar{\psi} i \gamma^\mu)(-iqe)\psi(x) = qe(\bar{\psi}\gamma^\mu\psi) \longrightarrow \boxed{J^\mu(x) = qe\bar{\psi}\gamma^\mu\psi}$$

and this is conserved on the equation of motion. In the free Dirac theory we have, indeed:

$$\partial_\mu J^\mu = qe\left[(\partial_\mu \bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)\right]$$

using the E.O.M. $(i\gamma^\mu\partial_\mu - m)\psi = 0$ and $\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0$:

$$\partial_\mu J^\mu = eq\left[im\bar{\psi}\psi + \bar{\psi}(-im\psi)\right] = 0$$

Consequently we are tempted to write:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - qeA_\mu\bar{\psi}\gamma^\mu\psi}$$

There is however a small subtelty: is the current $J^\mu = qe\bar{\psi}\gamma^\mu\psi$ still conserved in the new theory? We checked that J^μ is conserved in the free Dirac theory, but it is still conserved in the interacting theory? Let's check it:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{\partial \mathcal{L}}{\partial \psi} \longrightarrow \partial_\mu(\overline{\Psi} i \gamma^\mu) = -m\overline{\Psi} + qe A_\mu \overline{\Psi}\gamma^\mu \quad \text{that is} \quad \begin{cases} i(\partial_\mu \overline{\Psi})\gamma^\mu = -m\overline{\Psi} + qe A_\mu \overline{\Psi}\gamma^\mu \\ (-i)\gamma^\mu(\partial_\mu \psi) = -m\psi + qe A_\mu \gamma^\mu \psi \end{cases}$$

So the e.o.m. changed. However:

$$\partial_\mu J^\mu = qe\left[(\partial_\mu \overline{\Psi})\gamma^\mu \psi + \overline{\Psi}\gamma^\mu(\partial_\mu \psi)\right] = qe\left[\cancel{im\overline{\Psi}\psi} - iqe\cancel{A_\mu \overline{\Psi}\gamma^\mu\psi} + \cancel{im\overline{\Psi}\psi} + iqe\cancel{\overline{\Psi}A_\mu \gamma^\mu\psi}\right] = 0 \qquad \square$$

This is however a peculiarity of QED theory rather than a generic result.

EXAMPLE: Let us consider, for instance, the case of charged scalar particles (instead of fermions). Suppose we want to describe charged scalar particles with charge qe. Such particle from the QFT perspective are described by:

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left(a(\vec{p})e^{-ip\cdot x} + b^\dagger(\vec{p})e^{ip\cdot x}\right)$$
$$\phi^\dagger(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left(a^\dagger(\vec{p})e^{ip\cdot x} + b(p)e^{-ipx}\right)$$

From a Lagrangian perspective:

$$\boxed{\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi}$$

which allows us to find the e.o.m.:

$$(\square + m^2)\phi(x) = 0$$
$$(\square + m^2)\phi^\dagger(x) = 0.$$

The Lagrangian is invariant under the global symmetry:

$$\begin{cases} \phi(x) \longrightarrow \exp(-iqe\theta)\phi(x) & \longrightarrow D\phi = -iqe\phi \\ \phi^\dagger(x) \longrightarrow \exp(+iqe\theta)\phi^\dagger(x) & \longrightarrow D\phi^\dagger = +iqe\phi^\dagger \end{cases}$$

The Noether current is:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} D\phi^a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} D\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} D\phi^\dagger = (\partial^\mu \phi^\dagger)(-iqe\phi) + iqe\phi^\dagger(\partial^\mu \phi)$$

$$\longrightarrow \boxed{J^\mu = iqe\left[\phi^\dagger(\partial^\mu \phi) - (\partial^\mu \phi^\dagger)\phi\right]}$$

which is obvious conserved (in the free case):

$$\partial_\mu J^\mu = iqe\left[\cancel{(\partial_\mu \phi^\dagger)(\partial^\mu \phi)} + \phi^\dagger \square \phi - (\square \phi^\dagger)\phi - \cancel{(\partial^\mu \phi^\dagger)(\partial_\mu \phi)}\right] = iqe\left[\cancel{-m^2\phi^\dagger\phi} + \cancel{m^2\phi^\dagger\phi}\right] = 0$$

So we would be tempted to write the coupling:

$$\mathcal{L} \overset{?}{=} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2\phi^\dagger\phi + iqe A_\mu\left[\phi^\dagger(\partial^\mu \phi) - (\partial^\mu \phi^\dagger)\phi\right]$$

but it does not work because J^μ it is no longer conserved in the interacting theory. We could check it computing firstly the e.o.m. in the interacting theory and using them to prove explicitly that $\partial_\mu J^\mu \neq 0$.

We need a more general prescription : consider a theory described by the Lagrangian density :

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, A_\mu, \partial_\mu A_\nu)$$

and consider the case in which this theory enjoys a global U(1) symmetry

$$\phi(x) \longrightarrow \phi'(x) = e^{-iqe\theta}\phi(x)$$

By definition we have that $D\mathcal{L}=0$ and we have the conserved current :

$$J^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} D\phi$$

of course we don't Know its explicit expression because we don't Known the explicit form of $\mathcal{L}$. Consider the case in which $\theta = \theta(x)$. In the Jargon we say : "we "Gauge" the global symmetry ". Clearly it is no longer a symmetry of $\mathcal{L}$:

$$\phi(x) \longrightarrow \exp(-iqe\,\theta(x))\,\phi(x) \overset{\text{educated guess}}{\Longrightarrow} D\mathcal{L} = (\partial_\mu\theta)\,h^\mu(x)$$

→ $\mathcal{L}$ must be a scalar I need to contract with a 4-vector object

We could show that the true identification is :

$$\boxed{D\mathcal{L} = (\partial_\mu\theta(x))\,J^\mu(x)}$$

→ it is the conserved current defined before

The Hamilton principle of stationary says that the time evolution of the field configuration corresponds to a stationary action with respect to arbitrary variation of the fields (which vanish at large space time distances). This must be true for arbitrary variations, including the one we're considering here $\phi(x) \longrightarrow \phi(x) - iqe\,\theta(x)\phi(x)$. $\rightarrow D\phi = -iqe\,\theta(x)\,\phi(x)$.
If we sit on the EOM for $\phi(x)$ we must have that :

$$\delta S = \int d^4x\,(\partial_\mu\theta)\,J^\mu(x) = -\int d^4x\;\theta\,(\partial_\mu J^\mu) = 0$$

Now, suppose of introducing another "piece" : let's combine the transformations

$$\begin{cases} \phi(x) \longrightarrow e^{-iqe\theta(x)}\,\phi(x) \\ A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\theta(x) \end{cases} \quad ★$$

This implies that :

$$D\mathcal{L} = (\partial_\mu\theta)J^\mu + \frac{\partial \mathcal{L}}{\partial A_\mu}(\partial_\mu\theta) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\partial_\mu\partial_\nu\theta \overset{!}{=} 0$$

we impose that $D\mathcal{L} \overset{!}{=} 0$ because we want that $\mathcal{L}$ has to be invariant under this combined Gauge transformation . I get 2 conditions.

★ $$\boxed{J^\mu + \frac{\partial\mathcal{L}}{\partial A_\mu} = 0 \longrightarrow \frac{\partial\mathcal{L}}{\partial A_\mu} = -J^\mu} \longrightarrow$$

it tells us precisely what we want: A_μ is coupled with a conserved current

$$\boxed{\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}} \longrightarrow$$

it can be satisfied if the dependence from $\partial_\mu A_\nu$ enters in $\mathcal{L}$ via the anti-symm. combination $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$

We now have a clear prescription to build up our theory . We just need the following rules: The Lagrangian must be invariant under the combined Gauge transformation ★

The simplest way to achieve this invariance is called minimal coupling prescription; it tells us that if we start from a free theory and promote ∂_μ to be a covariant derivative then the theory will satisfy this invariance.

$$\partial_\mu \phi \longrightarrow \boxed{D_\mu \phi \equiv \partial_\mu \phi + ieq A_\mu \phi} \qquad \text{Covariant derivative}$$

COMMENT :

The combined Gauge transformation

$$\boxed{\begin{array}{l} \phi(x) \longrightarrow e^{-iqe\theta(x)} \phi(x) \\ A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \theta(x) \end{array}}$$

is not a symmetry since it does not reflect a change in the physical states on the system. However the way we derived here seems to contradict the above statement. After all, we impose the condition $D\mathcal{L} \overset{!}{=} 0$ which looks like the way we defined a symmetry in the field theory context.

We could certainly try to construct a conserved current by mimicking Noether theorem:

$$J^\mu_\theta \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \theta D\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} DA_\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \theta(-iqe)\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} (\partial_\nu \theta) = -\theta J^\mu(x) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} (\partial_\nu \theta)$$

$$\longrightarrow \boxed{J^\mu_\theta \equiv -\theta J^\mu(x) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} (\partial_\nu \theta)}$$

And we can see that it is indeed conserved:

$$\partial_\mu J^\mu_\theta = -(\partial_\mu \theta) J^\mu - \underset{\substack{\parallel \\ 0}}{\cancel{\theta(\partial_\mu J^\mu)}} \ \text{since } \partial_\mu J^\mu = 0 \ - \underbrace{\left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right]}_{= \frac{\partial \mathcal{L}}{\partial A_\nu}} (\partial_\nu \theta) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \partial_\mu \partial_\nu \theta =$$

$$= -(\partial_\mu \theta) J^\mu - \frac{\partial \mathcal{L}}{\partial A_\nu} (\partial_\nu \theta) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \partial_\mu \partial_\nu \theta = 0$$

As a consequence of ★

However, the Euler-Lagrange e.o.m. for A_μ allows to write:

$$J^\mu_\theta = \theta(x) \frac{\partial \mathcal{L}}{\partial A_\mu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \partial_\nu \theta = +\theta(x) \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} (\partial_\nu \theta) = +\partial_\nu \left[\theta(x) \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right]$$

$$\longrightarrow \boxed{J^\mu_\theta = \partial_\nu \left[\theta(x) \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right]}$$

Let's try to construct the conserved charge associate to it:

$$\boxed{Q_\theta \overset{?}{=} \int d^3\vec{x}\, J^0_\theta}$$

Let us give a closer look to J^0_θ. We have:

$$J^0_\theta = \partial_\nu \left(\theta(x) \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_0)}\right) = \underset{\text{for antisymmetry}}{\cancel{\partial_0 \left(\theta(x) \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)}\right)}} + \partial_i \left(\theta(x) \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)}\right) \longrightarrow \boxed{J^0_\theta = \partial_i \left(\theta(x) \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)}\right)}$$

↓

it is a spatial divergence

Consequently the conserved charge would be:

$$Q_\theta = \int d^3\vec{x}\; \partial_i\left(\theta(x)\frac{\partial \mathcal{L}}{\partial(\partial_i A_0)}\right) = \int_S \theta\, \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)}\, \hat{n}_i \cdot ds$$

Were Q is the conserved charge associated to the global symmetry. :

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial A_\mu} = -\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \longrightarrow J^0 = -\partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)} \longrightarrow Q = -\int_S \frac{\partial \mathcal{L}}{\partial(\partial_i A_0)}\, \hat{n}_i \cdot ds$$

Since $\frac{dQ}{dt} = 0$ the flux at spatial ∞ of $\frac{\partial \mathcal{L}}{\partial(\partial_i A_0)} \longrightarrow const$

Consequently in order for Q_θ to be conserved we have 2 possibilities :

1) $\theta(x) \xrightarrow{x\to\infty} 0$ but in this case $Q_\theta = 0$

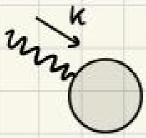
2) $\theta(x) \xrightarrow{x\to\infty} const \neq 0$ therefore $Q_\theta = \theta \cdot Q$ (it is not a new charge)

This is enough to conclude that we do not have a new symmetry besides the global one. To conclude the charge Q represents the generators of group G in the unitary representation according to which external states are classified. Gauging a global symmetry does change the Noether-s charge Q.

WARD IDENTITY

As we have discussed in the description of massless spin-1 field with the field $A^\mu(x)$ we are forced to declare that $\varepsilon_\mu(\vec{p},\lambda)$ and $\varepsilon_\mu(\vec{p},\lambda) + c_\lambda p_\mu$ describe the same physics. Conceptually this redundancy reflected into the fact that the physical photon field does not transform as a 4-vector: $A^\mu(x) \longrightarrow \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) + \partial^\mu \Omega(x)$. And in turn, the construction of a consistent interacting theory forced us to couple A_μ only with a conserved current.

Suppose we want to compute an amplitude where there is some photon involved (e.g. incoming γ)

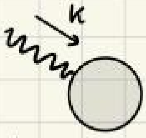
$\mathcal{M} =$ 

$= \varepsilon_\mu(\vec{k},\lambda)\, i\mathcal{M}^\mu$

(n.b. for an outgoing γ : $\varepsilon^*_\mu(\vec{k},\lambda)\, i M^\mu$

we isolate the contribution of some external γ with momentum $\vec{k}$ and helicity λ.

Now we remember that $\varepsilon^\mu(\vec{p},\lambda) \longleftrightarrow \varepsilon^\mu(\vec{k},\lambda) + c_\lambda k^\mu$:

$\longrightarrow \mathcal{M} =$ 

$= (\varepsilon_\mu + c\,k_\mu)\mathcal{M}^\mu \xrightarrow{\text{since the amplitude must be invariant}} \boxed{k_\mu \mathcal{M}^\mu = 0}$ Ward identity

Show if this is true is an advanced task, it needs path integral formulation of QFT.
This implies that using ε^μ instead of $\varepsilon^\mu + c\,k^\mu$ we'll have no difference.
The fact that this Ward identity is true is a consequence of the fact tha the photon field is coupled to a conserved current.

EXERCISE

Consider the Lagrangian of SQED : take $\mathcal{L} = \partial_\mu\phi^\dagger \partial^\mu\phi - m^2\phi^\dagger\phi$ and use minimal substit.

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - m^2\,\phi^\dagger\phi} \quad \text{with} \quad (D_\mu\phi = \partial_\mu\phi + ieq A_\mu\phi)$$

take $q=-1$ and the particle associated π^- $(q=-1)$

1) Extract the Feynman rules

Let's extract the interaction part of the Lagrangian :

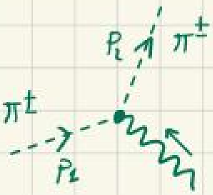
$$(D_\mu\phi)^\dagger(D^\mu\phi) = (\partial_\mu\phi + ieq A_\mu\phi)^\dagger(\partial^\mu\phi + ieq A^\mu\phi) =$$
$$= \underbrace{(\partial_\mu\phi^\dagger)\, ieq A^\mu\phi \; - ieq A_\mu\phi^\dagger(\partial^\mu\phi) + q^2e^2A_\mu A^\mu\phi^\dagger\phi}_{\text{interactions}} + \partial_\mu\phi^\dagger\partial^\mu\phi$$

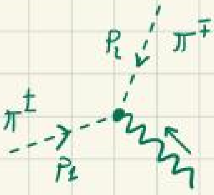
$$\longrightarrow \boxed{\mathcal{L}_{int} = ieq\, A^\mu\left[(\partial_\mu\phi^\dagger)\phi - \phi^\dagger(\partial_\mu\phi)\right] + q^2e^2A_\mu A^\mu\phi^\dagger\phi}$$

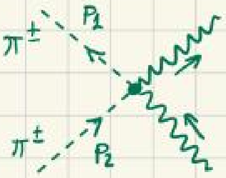
The field ϕ and its $\partial_\mu\phi$ is :

$$\boxed{\begin{aligned} \phi &= \int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[a(\vec{p})e^{-ipx} + b^\dagger(\vec{p})\,e^{ip\cdot x}\right] \\ \partial_\mu\phi &= \int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\,E_p}}\left[a(\vec{p})(-ip_\mu)\,e^{-ip\cdot x} + (ip_\mu)\,b^\dagger(p)\,e^{ip\cdot x}\right]\end{aligned}}$$

we have the following interaction vertices:







Consider the 1st diagram. We need to compute

$$S_{fi} = \langle\pi^+(p_2)|(-i)\int d^4x\,(+ie)\,A_\mu\left[(\partial^\mu\phi^\dagger)\phi - \phi^\dagger(\partial^\mu\phi)\right]|\pi^+(p_1),\gamma(k,\lambda)\rangle =$$
$$= +e\int d^4x\,\langle\pi^+(p_2)|A_\mu(\partial^\mu\phi^\dagger)\phi\,|\pi^+(p_1),\gamma(k,\lambda)\rangle - \langle\pi^+(p_2)|A_\mu\phi^\dagger(\partial^\mu\phi)|\pi^+(p_2),\gamma(k,\lambda)\rangle =$$

We now use that:

$$|\pi^+(p_1)\rangle = \sqrt{2E_{p_1}}\,b^\dagger(p_1)|0\rangle \qquad \langle\pi^+(p_2)| = \sqrt{2E_{p_2}}\,\langle 0|b(p_2)$$

so we only have the possibilities to make this applications:

$$(\partial^\mu\phi^\dagger)\,|\pi^+(p_1)\rangle = (-ip_1^\mu)\,e^{-ip_1\cdot x}|0\rangle \qquad (\text{since } \partial^\mu\phi^\dagger \text{ it contains } b)$$
$$\phi^\dagger\,|\pi^+(p_1)\rangle = e^{-ip_1\cdot x}|0\rangle$$

$$\langle\pi^+(p_2)|\,\phi = \langle 0|\,e^{ip_2\cdot x}$$
$$\langle\pi^+(p_2)|\partial^\mu\phi = \langle 0|(ip_2^\mu)\,e^{ip_2\cdot x}$$

So we have that:

$$S_{fi} = e\int d^4x \left[\varepsilon_\mu(\vec{k},\lambda)\, e^{-ik\cdot x} (-i P_1^\mu) e^{-iP_1\cdot x} e^{iP_2\cdot x} - \varepsilon_\mu(\vec{k},\lambda)\, e^{-ik\cdot x} (i P_2^\mu)\, e^{iP_2\cdot x} e^{-iP_1\cdot x} \right] =$$

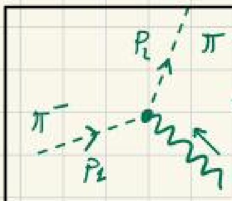
$$= (2\pi)^4\, \delta(-k + P_2 - P_1)\, \varepsilon_\mu(\vec{k},\lambda)\, (-iP_1^\mu - iP_2^\mu)\, e \quad = (-ie)\,(P_1+P_2)^\mu \cdot (2\pi)^4\, \delta(-k+P_2-P_1)\, \varepsilon_\mu(\vec{k},\lambda)$$

We strip off the polarization vector and the delta function to write:

- 

$\equiv (-ie)\,(P_1+P_2)^\mu$

or in a similar way we can get also that:



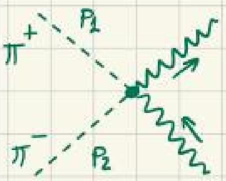
$\equiv ie\,(P_1+P_2)^\mu$

In a similar way we could extract the other Feynman rules:

- 

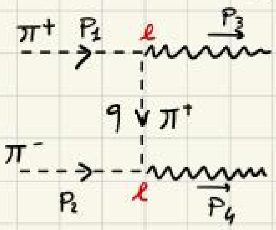
$\equiv ie\,(P_1-P_2)^\mu$

$\equiv ie\,(-P_1+P_2)^\mu$

- 

$\equiv i q^2 e\, g_{\mu\nu} \cdot 2$

2) Consider the amplitude of the process $\pi^+(P_1) + \pi^-(P_2) \longrightarrow \gamma(P_3,\lambda_3) + \gamma(P_4,\lambda_4)$.
We have the following diagrams:



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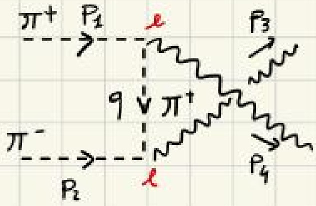

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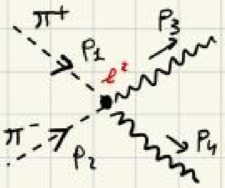


they contribut all to the 2nd order $(\sim e^2)$ so we have to consider all of them:

$$\longrightarrow \boxed{i\mathcal{M} = i\mathcal{M}_t + i\mathcal{M}_u + i\mathcal{M}_4}$$

- $i\mathcal{M}_t = (-ie)(P_1+q)^\mu \varepsilon_\mu^*(P_3) \dfrac{i}{q^2 - m^2 + i\varepsilon} (ie)(P_2-q)^\nu \varepsilon_\nu^*(P_4) =$

$$= \frac{ie^2}{q^2-m^2+i\varepsilon} (2P_1 - P_3)^\mu (2P_2-P_4)^\nu \varepsilon_\mu^*(P_3)\, \varepsilon_\nu^*(P_4) = \qquad P_1 = q + P_3 \longrightarrow q = P_1 - P_3$$

$$= \frac{ie^2}{(P_1-P_3)^2 - m^2} (2P_1-P_3)^\mu (2P_2-P_4)^\nu \varepsilon_\mu^*(P_3)\, \varepsilon_\nu^*(P_4)$$

- $i\mathcal{M}_u = (-ie)(P_1+q)^\mu \varepsilon_\mu^*(P_4) \dfrac{i}{q^2-m^2+i\varepsilon} (ie)(P_2-q)^\nu \varepsilon_\nu^*(P_3) = \qquad P_3 = q + P_2 \longrightarrow q = P_3 - P_2$

$$= \frac{ie^2}{(P_3-P_2)^2 - m^2} (2P_1 - P_4)^\mu (2P_2-P_3)^\nu \varepsilon_\mu^*(P_4)\, \varepsilon_\nu^*(P_3$$

- $i\mathcal{M}_4 = 2ie^2 g_{\mu\nu}\, \varepsilon^{\mu *}(P_3)\, \varepsilon^{\nu *}(P_4)$

We consider on shell scalar particles $P_1^2 = P_2^2 = m^2$. So:

$$\boxed{\begin{aligned} i\mathcal{M}_t &= \frac{ie^2}{m^2 + \cancel{m^2} - 2P_1\cdot P_3 - \cancel{m^2}}(2P_1 - P_3)^\mu (2P_2 - P_4)^\nu \varepsilon_\mu^*(P_3)\, \varepsilon_\nu^*(P_4) \\ i\mathcal{M}_u &= \frac{ie^2}{m^2 + \cancel{m^2} - 2P_1\cdot P_3 - \cancel{m^2}}(2P_1 - P_4)^\mu (2P_2 - P_3)^\nu \varepsilon_\mu^*(P_4)\, \varepsilon_\nu^*(P_3) \\ i\mathcal{M}_4 &= 2ie^2 g_{\mu\nu}\, \varepsilon^{\mu *}(P_3)\, \varepsilon^{\nu *}(P_4) \end{aligned}}$$

3) <u>Check the Ward identity</u> $P_\mu \mathcal{M}^\mu = 0$

We substitute $\varepsilon_\mu^*(P_3) \to P_3$. Let's take firstly the part without $\varepsilon_\mu^*(P_3)$:

- $$i\mathcal{M}_t^\mu = \frac{ie^2}{-2P_1\cdot P_3 + m^2}(2P_1 - P_3)^\mu (2P_2 - P_4)^\nu \varepsilon_\nu^*(P_4)$$

$$\longrightarrow i\mathcal{M}_t^\mu P_{\mu_3} = \frac{ie^2}{-2P_1\cdot P_3 + m^2}(2P_1\cdot P_3 - m^2)(2P_2 - P_4)^\nu \varepsilon_\nu^*(P_4) = (-ie^2)(2P_2 - P_4)^\nu \varepsilon_\nu^*(P_4)$$

- $$i\mathcal{M}_u^\mu = \frac{ie^2}{-2P_2\cdot P_3 + m^2}(2P_1 - P_4)^\mu (2P_2 - P_3)^\nu \varepsilon_\nu^*(P_4)$$

$$\longrightarrow i\mathcal{M}_u^\mu P_{\mu_3} = \frac{ie^2}{-2P_2\cdot P_3 + m^2}(2P_2\cdot P_3 - m^2)(2P_1 - P_4)^\nu \varepsilon_\nu^*(P_4) = (-ie^2)(2P_1 - P_4)^\nu \varepsilon_\nu^*(P_4)$$

- $$i\mathcal{M}_4^\mu = 2ie^2 g_{\mu\nu}\, \varepsilon_\nu^*(P_4)$$

$$\longrightarrow i\mathcal{M}_4^\mu P_{\mu_3} = 2ie^2 P_3^\mu \varepsilon_\nu^*(P_4)$$

Therefore :

$$\begin{aligned} i\mathcal{M}^\mu P_{3_\mu} &= (i\mathcal{M}_t^\mu + i\mathcal{M}_u^\mu + i\mathcal{M}_4^\mu) P_{3_\mu} = \\ &= ie^2 (-2P_2 + P_4 - 2P_1 + P_4 + 2P_3)^\mu \varepsilon_\mu^*(P_4) = \\ &= ie^2 (-2)\underbrace{(P_2 + P_1 - P_4 - P_3)^\mu}_{=0 \text{ (for momentum conservation}} \varepsilon_\mu^*(P_4) = 0 \end{aligned} \longrightarrow \boxed{i\mathcal{M}^\mu P_{3,\mu} = 0}$$

PHOTON PROPAGATOR

Consider the theory : $\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \quad ; \quad \partial_\mu J^\mu = 0}$

This theory is in perfect agreement with Maxwell

$$\longrightarrow \quad \boxed{\partial_\mu F^{\mu\nu} = J^\mu} \quad \text{Inhomogeneus Maxwell eq.}$$

If we write:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \; ; \; A^\mu = (\phi, \vec{A}) \qquad J^\mu = (\rho, \vec{J})$$

we find exactly the Maxwell equations, including the Gauge freedom $A'^\mu = A^\mu + \partial^\mu \alpha$
The theor propagates precisely 2 d.o.f.

i) If we write the e.o.m. in terms of A^μ, we get:

$$\partial_\mu F^\mu = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = (\partial_t^2 - \vec{\nabla}^2)A^\nu - \partial^\nu(\partial_t A^0 - \vec{\nabla}\cdot\vec{A}) = J^\mu$$

$$\longrightarrow \boxed{(\partial_t^2 - \vec{\nabla}^2)A^\nu - \partial^\nu(\partial_t A^0 - \vec{\nabla}\cdot\vec{A}) = J^\mu}$$

- $\nu = 0$: $\boxed{-\vec{\nabla}^2 A^0 - \partial_t(\vec{\nabla}\cdot\vec{A}) = J^0} \xrightarrow{J^0 = \rho,\ \vec{E} = -\vec{\nabla}\phi - \partial_t \vec{A}} \boxed{\vec{\nabla}\cdot\vec{E} = \rho}$ Gauss law

  (A^0 appears without $\partial_t A^0$)

  Consequently A^0 is not a dynamical variable, rather an aux. field enforcing the Gauss law as a time-independent constraint.

ii) The number of propagating d.o.f. is further deduced by the Gauge condition. In fact we have the freedom of choosing the arbitrary scalar function $\alpha(x)$ without changing the physics. Typically we exploit this freedom by taking a specific choice of α that enforces a specific constraint on A^μ. This procedure is called Gauge fixing. Typically choices are:

$$\boxed{\begin{array}{ll} \partial_\mu A^\mu = 0 & \text{(Lorentz or Landau Gauge)} \\ \vec{\nabla}\cdot\vec{A} = 0 & \text{(Coulomb Gauge)} \\ A^3 = 0 & \text{(Axial Gauge)} \end{array}}$$

On the QFT side the presence of redundancy $A_\mu \longrightarrow A_\mu + \partial_\mu \alpha$ in the Lagrangian : $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$. manifests in the fact that the photon propagator is not defined
By definition the propagator can be computed as the Green's function of the e.o.m.

$$\boxed{(\Box_x g^{\mu\nu} - \partial_x^\mu \partial_x^\nu) G_{\nu\rho}(x,y) = i\,\delta^\mu_\rho\, \delta(x-y)}$$

Going to Fourier space:

$$\begin{cases} G_{\nu\rho}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}\, \tilde{G}_{\nu\rho}(p) \\ \delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \end{cases} \longrightarrow \boxed{(-p^2 g^{\mu\nu} + p^\mu p^\nu)\tilde{G}_{\nu\rho}(p) = i\,\delta^\mu_\rho}$$

the idea is to invert that eq. However this is not possible because the operator $(-p^2 g^{\mu\nu} + p^\mu p^\nu)$ has zero eigenvalue being $(-p^2 g^{\mu\nu} + p^\mu p^\nu)p_\nu = -p^2 p^\mu + p^\mu p^2 = 0$.
The problem is solved once we pick up a Gauge . We consider the Lorentz Gauge: $\partial_\mu A^\mu = 0$.
We implement it in a rather fancy way , Known as R_ξ - Gauge .

Consider the Lagrangian: $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - A_\mu J^\mu}$

$-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ is the Gauge fixing term (ξ is a Lagrange multiplier) that enforces the specific Gauge that we want. Reapiling now the computation of the propagator since the e.o.m. is changed in the Maxwell eq in the chosen Gauge

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} - \frac{1}{2\xi} \frac{\partial}{\partial(\partial_\mu A_\nu)} (g^{\mu\nu} \partial_\mu A_\nu)^2 = -F_{\mu\nu} - \frac{1}{\xi} g^{\mu\nu} (\partial_\rho A^\rho)$$

$$\longrightarrow \partial_\mu \left(-\partial^\mu A^\nu + \partial^\nu A^\mu - \frac{1}{\xi} g^{\mu\nu} (\partial_\rho A^\rho) \right) = -J^\nu$$

$$\longrightarrow -\Box A^\nu + \partial^\nu (\partial_\mu A^\mu) - \frac{1}{\xi} \partial^\nu (\partial_\rho A^\rho) = -J^\nu$$

$$\longrightarrow \boxed{\Box A^\nu - \left(1 - \frac{1}{\xi}\right) \partial^\nu (\partial_\mu A^\mu) = J^\mu}$$

And the Green's function solves the equation:

$$\left[\Box_x g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial_x^\nu \partial_x^\mu \right] G_{\nu\rho}(x-y) = i \delta^\mu_\rho \, \delta(x-y)$$

that in Fourier space it reads:

$$\boxed{\left[-p^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p^\mu p^\nu \right] \tilde{G}_{\nu\rho}(p) = i \delta^\mu_\rho}$$

This equation now can be solved. We just impose the relativistic covariance to write the Ansatz

$$\tilde{G}_{\nu\rho}(p) = A(p^\nu) g_{\nu\rho} + B(p^\nu) p_\nu p_\rho \qquad \text{and solve for } A \text{ and } B$$

- $\left(-p^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p^\mu p^\nu\right)\left(A g_{\nu\rho} + B p_\nu p_\rho\right) = -p^2 A \delta^\mu_\rho - \cancel{p^2 B p^\mu p_\rho} + A\left(1 - \frac{1}{\xi}\right) p^\mu p_\rho + \left(\cancel{1} - \frac{1}{\xi}\right) B p^2 p^\mu p_\rho \overset{!}{=} i \delta^\mu_\rho$

$$\longrightarrow \begin{cases} -p^2 A = i \longrightarrow \boxed{A = -\frac{i}{p^2}} \\ p^\mu p_\rho \left[-\frac{i}{p^2}\left(1 - \frac{1}{\xi}\right) - \frac{1}{\xi} B p^2 \right] = 0 \longrightarrow \boxed{B = \frac{i}{p^4}(1-\xi)} \end{cases}$$

$$\longrightarrow \tilde{G}_{\nu\rho}(p) = \frac{-i}{p^2} g_{\nu\rho} + \frac{i}{p^4}(1-\xi) p_\nu p_\rho = \frac{+i}{p^2}\left[-g_{\nu\rho} + (1-\xi)\frac{p_\nu p_\rho}{p^2} \right]$$

$$\longrightarrow \boxed{\tilde{G}_{\nu\rho}(p) \equiv \frac{i}{p^2}\left[-g_{\nu\rho} + (1-\xi)\frac{p_\nu p_\rho}{p^2} \right]}$$

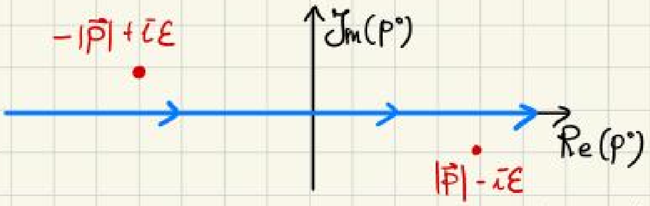
As usual this is not the end of the computation because we are actually interested in:

$$G_{\nu\rho}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \tilde{G}_{\nu\rho}(p)$$

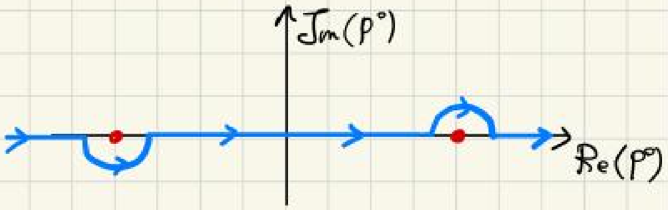
once we integrate over p there are singularities we must avoid. The strategy is to integrate first over p^0:

$$\int \frac{d^3\vec{p}}{(2\pi)^4} \int_{-\infty}^{+\infty} dp^0 \, e^{-ip^0(x^0-y^0) + i\vec{p}\cdot(\vec{x}-\vec{y})} \tilde{G}_{\nu\rho}(p)$$

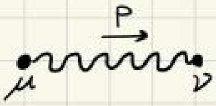
and avoid the pole at $p^0 = \pm|\vec{p}|$ with a specific regularization. The choice corresponding to the Feynman propagator is the so called time ordered Green's function:



or equivalently you keep the poles at $\pm|\vec{p}|$ but deform the contour of integration.



All in all, we arrive at the photon propagator in R_ξ gauge:



$$\equiv \frac{i}{p^2+i\varepsilon}\left[-g^{\mu\nu}+(1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

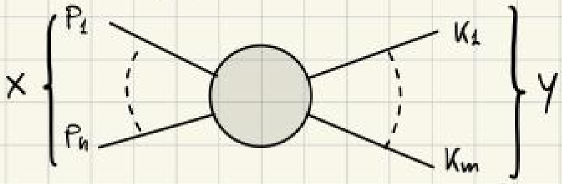
APPLICATIONS

1) THE WEINBERG SOFT THEOREM FOR PHOTONS

Consider a theory in which the photon interacts with multiple charged particles. We write:

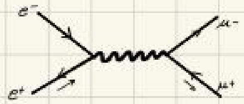
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_k \overline{\Psi}_k\left[i\gamma^\mu(\partial_\mu + ieq_kA_\mu) - m_k\right]\Psi_k$$

Consider the following process: $X \to Y$



$$\equiv i\mathcal{M}_0(p_1,\dots,p_n;k_1,\dots,k_m)$$

To fix ideas with a simple example think about $e^-(p_1)e^+(p_2) \longrightarrow \mu^+(k_1)\mu^-(k_2)$ described by the diagram:



at the tree level

We are more general: we consider charged initial and final states and, in addition, we consider a generic interaction "BLOB".

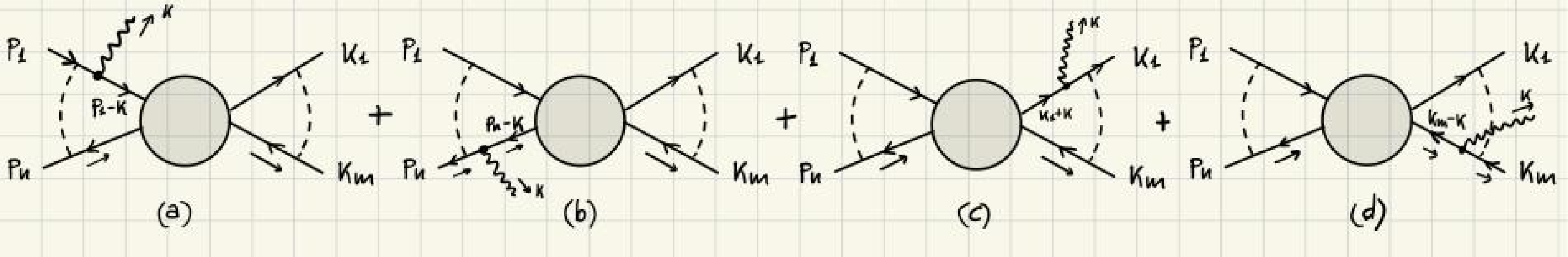
We now consider instead of the process $X \to Y$, the process:

$$\boxed{X \longrightarrow Y + \gamma(k)}$$

(In the final state we have an additional γ with $\vec{k}$)

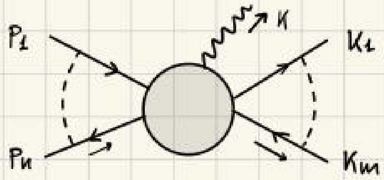
There are 2 classes of diagrams:

i) Emission from the external legs



(a) (b) (c) (d)

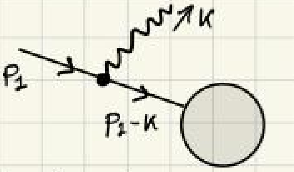
ii) Emission from the internal lines of the diagrams



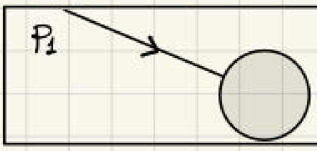
We now try to compute these diagrams under the assumption that the photon is soft. "Soft" in words mean that the energy of the emitted photon is small if compared to the energies / momenta of all the other particles partecipating to the process.

- DIAGRAM (i,a)

Let's focus on the leg that emit the photon

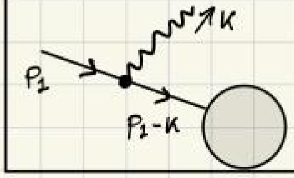


If the photon were absent this leg would contribute to the total amplitude as the spinor $u(p_1)$. Formally we write:



$$\equiv i\mathcal{M}_0 = i\overline{\mathcal{M}}_0(p_1,\dots,p_n;k_1,\dots,k_m)\,u(p_1)$$

However we now have an emission. Thus we write:



$$\equiv i\overline{\mathcal{M}}_0(p_1-k,p_2,\dots,p_n;k_1,\dots,k_m)\times\frac{i(\not{p}_1-\not{k}+m_1)}{((p_1-k)^2-m^2)}(-ieq_1^{in}\gamma^\mu)\varepsilon^*_\mu(k)\,u(p_1)$$

Let's focus on the factor:

$$\frac{eq_1^{in}(\not{p}_1-\not{k}_1+m_1)}{(\cancel{p_1^2}+\cancel{k^2}-2p_1\cdot k-\cancel{m^2})}\gamma^\mu\varepsilon^*_\mu(k)u(p_1) = \frac{eq_1^{in}(\not{p}_1-\not{k}+m_1)}{-2p_1\cdot k}\gamma^\mu\varepsilon^*_\mu(k)\,u(p_1) = (*)$$

$\overset{!!}{0}$ (photon) → $p_1^2 = m^2$ since the particle is on the mass shell

We now focus on the factor $(\not{p}_1-\not{k}+m_1)\gamma^\mu u(p_1)$. We use $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$ to write:

$$(\not{p}_1-k+m_1)\gamma^\mu u(p_1) = ((p_{1\nu}-k_\nu)\gamma^\nu-m_1)\gamma^\mu u(p_1) = ((p_1-k)_\nu\gamma^\nu\gamma^\mu+m_1\gamma^\mu)u(p_1) =$$

$$= ((p_1-k)_\nu(-\gamma^\mu\gamma^\nu+2g^{\mu\nu})+m_1\gamma^\mu)u(p_1) = \left[\gamma^\mu(-\not{p}_1+\not{k}+m_1)+2(p_1-k)^\mu\right]u(p_1) =$$

we use the Dirac eq: $(\not{p}_1-m_1)u(p_1)=0$

$$= (\gamma^\mu\not{k}+2(p_1-k)^\mu)u(p_1)$$

Therefore:

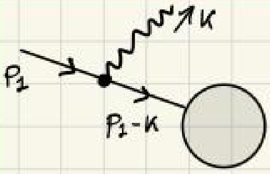
we use $\varepsilon^*_\mu(k)k^\mu = 0$

$$(*) = eq_1^{in}\varepsilon^*_\mu(k)\frac{\left[\gamma^\mu\not{k}+2(p_1-\cancel{k})^\mu\right]}{-2p_1\cdot k} = eq_1^{in}\varepsilon^*_\mu(k)\frac{(\gamma^\mu\not{k}+2p_1^\mu)}{-2p_1\cdot k}u(p_1) =$$

We now use the soft photon absumption and neglet the term prop. to the photon momentum K

$$\approx e\, q_1^{in} \varepsilon_\mu^*(K) \frac{\cancel{2} P_1^\mu}{-\cancel{2} P_1 \cdot K} u(P_1) = \frac{-e\, q_1^{in} P_1^\mu}{P_1 \cdot K} \varepsilon_\mu^*(K) u(P_1)$$

If we now come back to the original amplitude, we notice something interesting. Let's write it:



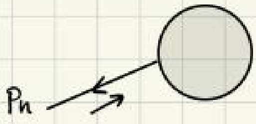
$$\equiv i \overline{M}_0 (\overbrace{P_1 - K}^{\approx P_1}, P_2, \dots, P_n; K_1, \dots, K_m) \times \left(\frac{-e\, q_1^{in} P_1^\mu}{P_1 \cdot K} \varepsilon_\mu^*(K) \right) u(P_1) \approx$$

$$\approx i M_0 \left(\frac{-e\, q_1^{in} P_1^\mu}{P_1 \cdot K} \varepsilon_\mu^*(K) \right)$$

in the soft limit we factorize the amplitude .
for the initial process. $X \longrightarrow Y$

- DIAGRAM (i, b)

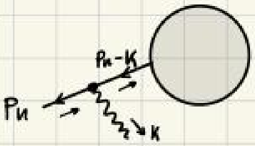
The logic of the computation is the same, there are only few thecnical differenes. We focus on one leg that emits the photon. We know that without emission the amplitude would be:



$$\equiv i M_0 = \overline{v}(P_n)\, i \tilde{M}(P_1, \dots, P_n; K_1, \dots, K_m)$$

incoming antifermion

Considering the emission, we write:



$$\equiv \overline{v}(P_n)(-i q_n^{in} e \gamma^\mu) \varepsilon_\mu^*(K) \frac{i(-\not{P}_n + \not{K} + m)}{((P_n - K)^2 - m^2)} \times \; i \tilde{M}(P_1, \dots, P_n - K; K_1, \dots, K_m)$$

We focus on the emission factor:

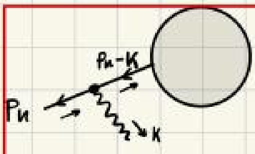
$$\overline{v}(P_n)\, q_n^{in} e\, \varepsilon_\mu^*(K) \frac{\gamma^\mu(-\not{P}_n + \not{K} + m_n)}{(\cancel{P_n^2} + \underset{\overset{\shortparallel}{0}}{\cancel{K^2}} - 2P_n \cdot K - \cancel{m^2})} = \frac{q_n^{in} e\, \varepsilon_\mu^*(K)}{-2P_n \cdot K} \overline{v}(P_n) \left[\gamma^\mu \gamma^\nu (-P_n + K)_\nu + \gamma^\mu m_n \right] =$$

$$= \frac{q_n^{in} e\, \varepsilon_\mu^*(K)}{-2P_n \cdot K} \overline{v}(P_n) \left[(\not{P}_n - \not{K} + m_n)\gamma^\mu + 2(-P_n + K)^\mu \right] =$$

using $\varepsilon_\mu^*(K) K^\mu = 0$
$\overline{v}(P_n)(\not{P}_n + m_n) = 0$

$$= \frac{q_n^{in} e\, \varepsilon_\mu^*(K)}{-2P_n \cdot K} \overline{v}(P_n) \left(-\not{K}\gamma^\mu - 2P_n^\mu \right) \approx \frac{+q_n^{in} e\, P_n^\mu \varepsilon_\mu^*(K)}{P_n \cdot K} \overline{v}(P_n)$$

Therefore:

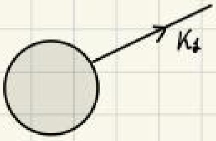


$$\approx \frac{+q_n^{in} e\, P_n^\mu \varepsilon_\mu^*(K)}{P_n \cdot K} \overline{v}(P_n)\; i \tilde{M}(P_1, \dots, \overbrace{P_n - K}^{\approx P_n}; K_1, \dots, K_m) \approx$$

$$\approx i M_0 \frac{q_n^{in} e\, P_n^\mu \varepsilon_\mu^*(K)}{P_n \cdot K}$$

- DIAGRAM (i, c)

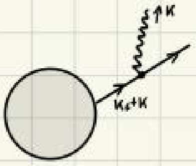
Without emission:



$$\equiv i M_0 = \overline{u}(K_1)\, i \tilde{M}_0(P_1, \dots, P_n; K_1, \dots, K_m)$$

outgoing fermion

Considering the emission, we write:



$$\equiv \bar{u}(k_1)(-iq_1^{fin}e\gamma^\mu)\,\varepsilon_\mu^*(k)\,\frac{i(\not{k}_1+\not{k}+m_1)}{((k_1+k)^2-m_1^2)}\times \quad i\tilde{\mathcal{M}}_0(p_1,\dots,p_n;k_1+k,\dots,k_m)$$

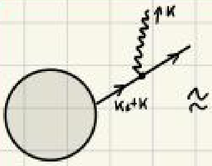
The manipulations are the usual ones in the prefactor:

$$\bar{u}(k_1)\,e\,q_1^{fin}\,\varepsilon_\mu^*(k)\,\gamma^\mu\,\frac{(\not{k}_1+\not{k}+m_1)}{\cancel{k_1^2}+\cancel{k^2}+2k_1\cdot k-\cancel{m^2}} = \frac{e\,q_1^{fin}\varepsilon_\mu^*(k)}{2k_1\cdot k}\,\bar{u}(k_1)\times\left[\gamma^\mu\gamma^\nu(k_1+k)_\nu+m_1\gamma^\mu\right]=$$

$$= \frac{e\,q_1^{fin}\varepsilon_\mu^*(k)}{2k_1\cdot k}\,\bar{u}(k_1)\times\left[(-\not{k}_1-\not{k}+m_1)\gamma^\mu+2(k_1+k)^\mu\right] = \longrightarrow \text{using } \bar{u}(k_1)(\not{k}_1-m_1)=0$$

$$\approx \frac{e\,q_1^{fin}\varepsilon_\mu^*(k)}{\cancel{2}k_1\cdot k}\,\bar{u}(k_1)\cdot\cancel{2}k_1^\mu$$

Therefore:

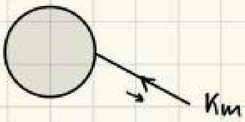


$$\approx \frac{e\,q_1^{fin}\varepsilon_\mu^*(k)}{k_1\cdot k}\,k_1^\mu\,\bar{u}(k_1)\;i\tilde{\mathcal{M}}_0(p_1,\dots,p_n;\overset{\approx k_1}{k_1+k},\dots,k_m)\approx$$

$$\approx i\mathcal{M}_0\cdot\frac{e\,q_1^{fin}\varepsilon_\mu^*(k)}{k_1\cdot k}\,k_1^\mu$$

- DIAGRAN (i,d)

Without emission:



$$\equiv i\mathcal{M}_0 = i\tilde{\mathcal{M}}_0(p_1,\dots,p_n;k_1,\dots,k_m)\,v(k_m)$$

Considering the emission:

$$\equiv i\tilde{\mathcal{M}}_0(p_1,\dots,p_n;k_1,\dots,k_m+k)\times\frac{i(-\not{k}_m-\not{k}+m)}{(+k_m+k)^2-m^2}\left(-ie\,q_m^{fin}\gamma^\mu\varepsilon_\mu^*(k)\right)v(k_m)$$

As usual we rewrite the prefactor as:

$$\frac{i(-\not{k}_m-\not{k}+m)(-ie\,q_m^{fin}\gamma^\mu\varepsilon_\mu^*(k))}{(\cancel{k_m^2}+\cancel{k^2}+2k_m\cdot k-\cancel{m^2})}\,v(k_m) = \frac{e\,q_m^{fin}\varepsilon_\mu^*(k)}{2k_m\cdot k}\left[(-k_m-k)_\nu\gamma^\nu\gamma^\mu+m\gamma^\mu\right]v(k_m)=$$

$$= \frac{e\,q_m^{fin}\varepsilon_\mu^*(k)}{2k_m\cdot k}\left[\gamma^\mu(\not{k}_m+\not{k}+m)+2(-k_m-k)^\mu\right]v(k_m) =$$

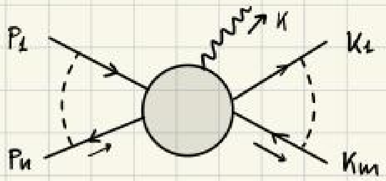
$$= \frac{e\,q_m^{fin}\varepsilon_\mu^*(k)}{2k_m\cdot k}\left[\gamma^\mu\not{k}-2k_m^\mu\right]v(k_m)\approx\frac{e\,q_m^{fin}\varepsilon_\mu^*(k)}{\cancel{2}k_m\cdot k}\left(-\cancel{2}k_m^\mu\right)v(k_m)$$

Therefore:

$$\approx i\tilde{\mathcal{M}}_0(p_1,\dots,p_n;k_1,\dots,\overset{\approx k_m}{k_m+k})\,v_{k_m}\left(\frac{-e\,q_m^{fin}\varepsilon_\mu^*(k)}{2k_m\cdot k}\right)\approx$$

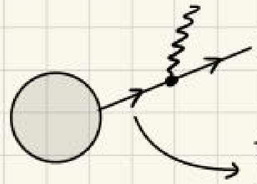
$$\approx i\mathcal{M}_0\left(\frac{-e\,q_m^{fin}\varepsilon_\mu^*(k)}{2k_m\cdot k}\right)$$

We should now include all diagrams with "internal emission of photons".



the point is that, in the soft limit, all these diagrams can be neglected. The point is indeed that these photons are always attacked to an internal leg of the diagram that is not on the mass shell (virtual).

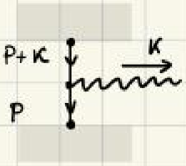
When the photon attached to an external leg:



$K_i^2 = m^2$ (on the mass shell, physica particle)

this intermediate propagator develops the behaviour $\frac{1}{K_i \cdot K}$ and dominates when $K \to 0$

Suppose now that the photon is emitted from an internal leg that is virtual:



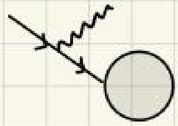
Consequently the denominator of the intermediate propagator is:

$$\frac{1}{((P+K)^2 - m^2)} = \frac{1}{P^2 + K^2 + 2P\cdot K - m^2} = \frac{1}{P^2 + 2P\cdot K - m^2} \overset{K\to 0}{\approx} \frac{1}{P^2 - m^2}$$

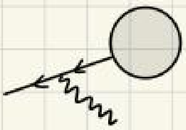
It is subleading compared to the behaviour $1/K_i \cdot K$ as $K \to 0$ of the other diagrams studied before

To sum up we found that the amplitude for the soft emission process $X \to Y + \gamma(K)$ takes a remarkably simple form. Before writing the general sum let's give a closer look to the first 2 results:

electric charge of the emitting particle



$$\approx iM_0(-1)(eq_1^{in})\frac{P_1 \cdot \mathcal{E}^*(K)}{P_1 \cdot K}$$



$$\approx iM_0(-1)(-eq_n^{in})\frac{P_n \cdot \mathcal{E}^*(K)}{P_n \cdot K}$$

electric charge of the emitting particle (the minus sign is because it is an antiparticle which is emitting the photon)

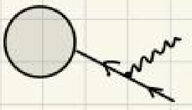
Let's just indicate with Q_i^{in}, $i=1,\dots,n$ the electric charge of the emitting particle from the i^{th} leg in the initial state. As a consequence the sum over all diagrams in which the photon is emitted from the initial state can be written as a single sum:

$$\boxed{iM_0 \sum_{i=1}^{n} (-1) Q_i^{in} \frac{P_i \cdot \mathcal{E}^*(K)}{P_i \cdot K}}$$

In a complete similar fashion we can write the sum over diagrams in which the photon is emitted by final-state legs. Let's have a look to the 3rd and 4th contribution:

electric charge of the emitting particle

$$\approx i\mathcal{M}_0 \left(e q_1^{fin}\right) \frac{k_1 \cdot \varepsilon^*(k)}{k_1 \cdot k}$$



$$\approx i\mathcal{M}_0 \left(-e q_m^{fin}\right) \frac{k_m \cdot \varepsilon^*(k)}{k_m \cdot k}$$

electric charge of the emitting particle (the minus sign is because it is an antiparticle which is emitting the photon)

Let's just indicate with Q_j^{fin}, $j=1,\dots,m$ the electric charge of the emitting particle from the j^{th} leg in the final state. As a consequence the sum over all diagrams in which the photon is emitted from the final state can be written as a single sum:

$$\boxed{i\mathcal{M}_0 \sum_{j=1}^{m} Q_j^{fin} \frac{k_j \cdot \varepsilon^*(k)}{k_j \cdot k}}$$

All in all, we arrive at our final expression:

$$\boxed{i\mathcal{M}(X \to Y + \gamma(k)) \approx i\mathcal{M}_0 \left[\sum_{j=1}^{m} Q_j^{fin} \frac{k_j \cdot \varepsilon^*(k)}{k_j \cdot k} - \sum_{i=1}^{n} Q_i^{in} \frac{p_i \cdot \varepsilon^*(k)}{p_i \cdot k} \right]}$$

The physical interpretation is the following, a soft photon has a very long wavelength ($k \to 0$) and, therefore, it does not care about physics at much shorter distances (higher energies)

Now the crucial point: the physics must remain the same if we shift $\varepsilon^\mu(k)$ by something prop. to k^μ. Consequently if we substitute $\varepsilon^\mu(k) \longleftrightarrow k^\mu$ we must get zero. If we impose this condition on our amplitude we find:

$$i\mathcal{M}(X \to Y + \gamma(k)) \approx i\mathcal{M}_0 \left[\sum_{j=1}^{m} Q_j^{fin} \frac{\cancel{k_j \cdot k}}{\cancel{k_j \cdot k}} - \sum_{i=1}^{n} Q_i^{in} \frac{\cancel{p_i \cdot k}}{\cancel{p_i \cdot k}} \right] \overset{!}{=} 0$$

$$\longrightarrow \boxed{\sum_{j=1}^{m} Q_j^{fin} \overset{!}{=} \sum_{i=1}^{n} Q_i^{in}}$$

TOTAL CHARGE CONSERVATION

It is worth pausing to link various pieces of information that we learned so far:

$$\boxed{\varepsilon^\mu(\vec{p},\lambda) = \varepsilon^\mu(\vec{p},\lambda) + c_\lambda p^\mu}$$

The physics must remain invariant under the shift since this shift is generated by a transformation under which phys. states do not transform.

$$\boxed{\sum_{j=1}^{m} Q_j^{fin} \overset{!}{=} \sum_{i=1}^{n} Q_i^{in}}$$

total charge conservation via Weinberg soft limit theorem

$$\boxed{\begin{matrix} A_\mu J^\mu \\ (\partial_\mu J^\mu = 0) \end{matrix}}$$

The condition $\varepsilon^\mu = \varepsilon^\mu + c_\lambda p^\mu$ is concretely realized by coupling A_μ with a conserved current J^μ

Symmetry a la Wigner-Weyl implies selection rules

$$\boxed{G = U(1)}$$

We realize the presence of a conserved current by imposing a U(1) symmetry realized a la Wigner-Weil

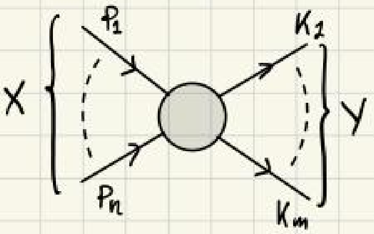
Generalization for gravitons

Quantum Gravity part

We introduced also the graviton field $h^{\mu\nu}(x)$ (massless spin-2 particle)

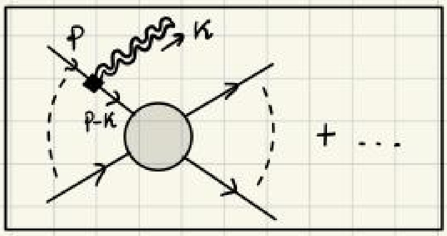
$$h^{\mu\nu}(x) = \sum_{\lambda=\pm 2} \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left[\varepsilon^{\mu\nu}(\vec{p},\lambda) a(\vec{p},\lambda) e^{-ip\cdot x} + h.c. \right]$$

we would like to understand something about possible interactions of this field with matter. First of all we consider a generic process $X \rightarrow Y$ of the kind:

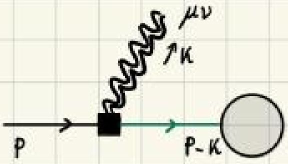


$$\equiv i\mathcal{M}_b(p_1,\dots,p_n;K_1,\dots,K_m)$$

and we consider the process $X \rightarrow Y + g(K)$ so with the emission of an extra soft massless particle with spin-2. We focus on the diagram with emission from external legs, the only ones that are enhanced by the soft pole when $K \rightarrow 0$



What is the structure of the interaction in the soft limit? Let's try to guess the emission factor:



$$\approx \Gamma^{\mu\nu}(p,K)\frac{1}{-2p\cdot K}$$

- from the denominator of this propagator we'll have the factor $\frac{1}{(p-K)^2-m^2} = \frac{1}{\cancel{p^2}+\cancel{K^2}-2p\cdot K-\cancel{m^2}} = \frac{1}{-2p\cdot K}$
- $\Gamma^{\mu\nu}$ is a tensor, and it must have a structure of the kind:

$$\Gamma^{\mu\nu}(p,K) = 2p^\mu p^\nu F(p^2,K^2,p\cdot K) + \underbrace{p^\mu q^\nu G(p^2,K^2,p\cdot K) + q^\mu p^\nu \tilde{G}(p^2,K^2,p\cdot K) + q^\mu q^\nu \bar{G}(p^2,K^2,p\cdot K)}_{\downarrow}$$

all of these structures don't contribute when contracted with $\varepsilon^{\mu\nu}(\vec{K},\pm 2) = \varepsilon^{\mu}(\vec{K},\pm 1)\varepsilon^{\nu}(\vec{K},\pm 1)$

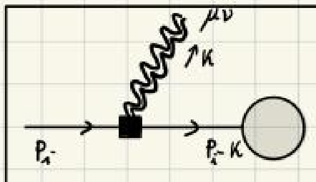
From the dimension analysis and taking into account that $K^2=0$, F can only depend on the dimensionless ratio $\frac{p\cdot K}{m^2}$:

$$\Gamma^{\mu\nu}(p,K) = 2p^\mu p^\nu F\left(\frac{p\cdot K}{m^2}\right)$$

Furthermore in the soft limit the leading term is the one with $K \rightarrow 0$:

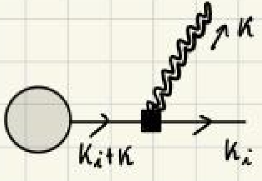
$$\Gamma^{\mu\nu}(p,K) \approx 2p^\mu p^\nu F(0)$$

In conclusion, if we consider the emission from an initial leg in the soft limit we write:



$$\approx i\mathcal{M}_0 \, (-1) \, \underbrace{F_i(0)}_{\equiv g_i} \, \frac{p_i^\mu p_i^\nu \varepsilon^*_{\mu\nu}(k)}{p_i \cdot k}$$

Similarly, if we consider emission from a final leg, we write:



$$\approx i\mathcal{M}_0 \cdot \underbrace{F_i(0)}_{\equiv g_i} \, \frac{k_i^\mu k_i^\nu \varepsilon^*_{\mu\nu}(k)}{k_i \cdot k}$$

As a consequence, for the amplitude of the soft emission process $X \to Y + g(k)$ we write:

$$\boxed{i\mathcal{M}(X \to Y + g(k)) \approx i\mathcal{M}_0 \left[\sum_{j=1}^{m} g_j \frac{k_j^\mu k_j^\nu \varepsilon^*_{\mu\nu}(k)}{k_j \cdot k} - \sum_{i=1}^{n} g_i \frac{p_i^\mu p_i^\nu \varepsilon^*_{\mu\nu}(k)}{p_i \cdot k} \right]}$$

Now the key point. As we have discussed, $\varepsilon^{\mu\nu}(\vec{k}, \lambda)$ and $\varepsilon^{\mu\nu}(\vec{k}, \lambda) + k^\mu v_\pm^\nu(k) + k^\nu v_\pm^\mu(k)$ must describe the same physics. This means that the shifting pieces should give a ϕ contribution to the amplitude. Consequently we obtain the condition:

$$\sum_{j=1}^{m} g_j \, k_j^\mu \frac{\cancel{k_j \cdot k}}{\cancel{k_j \cdot k}} - \sum_{i=1}^{n} g_i \, p_i^\mu \frac{\cancel{p_i \cdot k}}{\cancel{p_i \cdot k}} \overset{!}{=} 0 \longrightarrow \boxed{\sum_{\substack{\text{outgoing} \\ \text{particles}}} g_j k_j^\mu \overset{!}{=} \sum_{\substack{\text{ingoing} \\ \text{particles}}} g_i p_i^\mu} \quad (*)$$

There is something strange with this condition. The point is that momenta already verify the condition:

$$p_1 + \dots + p_n = k_1 + \dots + k_m$$

therefore we could fix one of the momenta in terms of the others. For ex: $p_1 = k_1 + \dots + k_m - (p_2 + \dots + p_n)$. However if we add a second constraint like the one in eq $(*)$, it would be possible to find a different solution for the same p_1, which is impossible. The only possibility is to have:

$$\boxed{g = g_i \quad \text{for all } i}$$

So we found that it is possible to construct an interacting theory for massless spin-2 particles only if their interactions with matter are universal. This is indeed what gravity does.

INTERACTIONS OF A MASSLESS SPIN-2 PARTICLE AT THE LAGRANGIAN LEVEL

Given the result of the previous section it is important to investigate what kind of interactions are described by a massless spin-2 field at the Lagrangian level. The situation is remarkably similar to that of the photon field. In fact, we have seen that the field $h^{\mu\nu}(x)$ does not transform under Lorentz as an antisymmetric rank-2 tensor. Specifically we found that:

$$U(\Lambda) \, h^{\mu\nu} \, U(\Lambda)^{-1} = (\Lambda^{-1})^\mu{}_\rho \, (\Lambda^{-1})^\nu{}_\sigma \, h^{\rho\sigma}(\Lambda x) + \partial^\mu \xi^\nu(\Lambda, x) + \partial^\nu \xi^\mu(\Lambda, x)$$

therefore at the Lagrangian level, we cannot write a term like $h_{\mu\nu}(x)\, h^{\mu\nu}(x)$ since, despte being "Lorentz invariant looking" it is not truly Lorentz invariant.
We need an additional condition. We require the following rule:

> The only terms admissible in the Lagrangian must be "Lorentz invariant looking" and also invariant under "double shift":
>
> $$h^{\mu\nu}(x) \longrightarrow h^{\mu\nu}(x) + \partial^\mu \xi^\nu(x) + \partial^\nu \xi^\mu(x)$$

There are 2 possibilities to implement this strategy:

1) In electromagnetism, one defines the object $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which is manifestly invariant under the shift $A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x)$ and construct terms in the Lagrangian directly of the type $F_{\mu\nu}F^{\mu\nu}$. Is there something similar also in the case of massless spin-2 ?

It is actually simple to construct something analogue to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Consider the combination

$$\boxed{R_{\mu\nu\rho\sigma} \equiv \frac{1}{2}\left(\partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\rho h_{\mu\sigma} - \partial_\mu \partial_\sigma h_{\nu\rho} + \partial_\nu \partial_\sigma h_{\mu\rho}\right)}$$

Linearized Riemann Tensor

This is obviously completely unaffected by the double shift. So one can then try to construct interactions directly from $R_{\mu\nu\rho\sigma}$.

2) The 2nd possibility, just like we did for photons, is to consider a coupling using directly $h_{\mu\nu}(x)$ of the form:

$$\boxed{h_{\mu\nu}(x)\, T^{\mu\nu}(x)}$$

with $T^{\mu\nu}$ a symmetric rank-2 tensor (so that $h_{\mu\nu}T^{\mu\nu}$ is "Lorentz inv. looking").
In order to be invariant under the double shift we must impose:

$$(\partial_\mu \xi_\nu) T^{\mu\nu} + (\partial_\nu \xi_\mu) T^{\mu\nu} \overset{!}{=} 0 \longrightarrow \boxed{2(\partial_\mu \xi_\nu) T^{\mu\nu} = 0}$$

At the level of the action, this condition is equivalent to the requirement that $T^{\mu\nu}$ is conserved.

We can therefore write an interaction of the kind:

$$\boxed{\mathcal{L}_{int} = \underset{\substack{\uparrow \\ \text{coupling in front} \\ \text{must be universal}}}{K}\; h_{\mu\nu}(x) \underset{\substack{\nwarrow \\ \text{we only know on conserved} \\ \text{symmetric rank-2 tensor}}}{T^{\mu\nu}(x)} \qquad ; \; \partial_\mu T^{\mu\nu} = 0}$$

the energy momentum tensor

Comments:

1) K is dimension full

- The field $h_{\mu\nu}$ has mass-dimension 1. In fact:

$$h_{\mu\nu}(x) = \sum_{\lambda} \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left[\varepsilon^{\mu\nu}(\vec{p},\lambda)\, e^{-ip\cdot x}\, a(\vec{p},\lambda) + h.c. \right] \longrightarrow [h_{\mu\nu}] = [M]^{\frac{5}{2}-\frac{3}{2}} = [M]$$

$[M]^{3-\frac{1}{2}} = [M]^{\frac{5}{2}}$

dimensionless

Since $[a, a^\dagger] = \delta(\vec{p}-\vec{k})(2\pi)^3$ and
$\int d^3\vec{p}\, \delta(\vec{p}) = 1 \longrightarrow [\delta] = [M]^{-3}$ and
we have then $[a] = [a^\dagger] = [M]^{-3/2}$

- $T^{\mu\nu}$ has mass-dim. 4 in fact $P^\mu = \int d^3\vec{x}\, T^{0\mu} \longrightarrow [T^{\mu\nu}] = \frac{[M]}{[V]} = [M]^4$

Since $[\mathcal{L}_{int}] \overset{!}{=} [M]^4$ it follows that:

$$\boxed{[\kappa] = [M]^{-1}}$$

So the universal coupling is dimensionfull and then the interaction is not renormalizible. (I would be tempted to take $\kappa = \sqrt{G_N} = \frac{1}{M_{plank}}$

(N.B. if we confront with E.M. $\mathcal{L}_{int} = -q e A_\mu J^\mu$; $J^\mu = \overline{\psi}\gamma^\mu\psi$)

2) Consider the canonical energy-momentum tensor that we derived:

$$\boxed{T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}(\partial^\nu\phi^A) - g^{\mu\nu}\mathcal{L} \quad ; \quad \partial_\mu T^{\mu\nu} = 0}$$

In general this object is <u>not symmetric</u>. This is simply to see. Consider:

$$\mathcal{M}^{\mu\rho\sigma} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}(-i)(J_S^{\rho\sigma})^A{}_B\,\phi^B + (x^\rho T^{\mu\nu} - x^\sigma T^{\mu\rho})$$

we rewrite:

$$\mathcal{M}^{\mu\rho\sigma} \equiv S^{\mu\rho\sigma} + x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

and now compute the 4-derivative:

using $\partial_\mu T^{\mu\nu} = 0$

$$\partial_\mu \mathcal{M}^{\mu\rho\sigma} = \partial_\mu S^{\mu\rho\sigma} + \delta^\rho_\mu T^{\mu\sigma} + \delta^\sigma_\mu T^{\mu\rho} =$$

$$= \partial_\mu S^{\mu\rho\sigma} + T^{\rho\sigma} + T^{\sigma\rho} \overset{!}{=} 0 \longrightarrow \boxed{T^{\sigma\rho} - T^{\rho\sigma} = \partial_\mu S^{\mu\rho\sigma}}$$

What we learn from this simple computation is that in general $T^{\mu\nu} \neq T^{\nu\mu}$. We also learn that the intrinsic spin of the fields is responsible for the canonical energy mom. tensor not being symmetric in general.
However, we also know that this is not a problem. We indeed know that we have the freedom of redefining the Nöther currents without altering the Nöther charges.
We define therefore:

$$\boxed{\Theta^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad ; \quad K^{\lambda\mu\nu} = -K^{\lambda\mu\nu}} \longrightarrow \text{does not change Nöther's charges.}$$

canonical energy momentum tensor

arbitrary rank-3 tensor antisymmetric in the first 2 indices

The explicit choice of $K^{\lambda\mu\nu}$ can be made by imposing the condition:

$$\boxed{\Theta^{\mu\nu} \overset{!}{=} \Theta^{\nu\mu}}$$

from which we get:

$$\textcircled{H}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \overset{!}{=} T^{\nu\mu} + \partial_\lambda K^{\lambda\nu\mu}$$

that we recast as follows:

$$\longrightarrow \quad T^{\mu\nu} - T^{\nu\mu} \overset{!}{=} \partial_\lambda \left(K^{\lambda\nu\mu} - K^{\lambda\mu\nu} \right)$$

$$\longrightarrow \quad \partial_\lambda S^{\lambda\nu\mu} \overset{!}{=} \partial_\lambda \left(K^{\lambda\nu\mu} - K^{\lambda\mu\nu} \right)$$

that can be solved if we set:

$$\boxed{S^{\lambda\nu\mu} = K^{\lambda\nu\mu} - K^{\lambda\mu\nu}}$$

with:

$$S^{\lambda\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi^A)} (-i) (J_s^{\nu\mu})^A{}_B \, \phi^B$$

<u>Exercise</u> : <u>*find an explicit expression for* $K^{\lambda\nu\mu}$</u>

We consider $S^{\lambda\nu\mu} = K^{\lambda\nu\mu} - K^{\lambda\mu\nu}$ and consider cyclic permutations of indices

$$S^{\mu\lambda\nu} = K^{\mu\lambda\nu} - K^{\mu\nu\lambda}$$

$$S^{\nu\mu\lambda} = K^{\nu\mu\lambda} - K^{\nu\lambda\mu}$$

we now add the first and last eq. and subtract the 2nd:

$$\begin{aligned} S^{\lambda\nu\mu} + S^{\nu\mu\lambda} - S^{\mu\lambda\nu} &= \left(K^{\lambda\nu\mu} - K^{\lambda\mu\nu} \right) + \left(K^{\nu\mu\lambda} - K^{\nu\lambda\mu} \right) - \left(K^{\mu\lambda\nu} - K^{\mu\nu\lambda} \right) = \\ &= \left(K^{\lambda\nu\mu} - K^{\nu\lambda\mu} \right) + \underbrace{\left(-K^{\lambda\mu\nu} - K^{\mu\lambda\nu} \right)}_{-K^{\lambda\mu\nu} + K^{\lambda\mu\nu} = 0} + \underbrace{\left(K^{\nu\mu\lambda} + K^{\lambda\nu\lambda} \right)}_{K^{\nu\mu\lambda} - K^{\nu\mu\lambda} = 0} = \\ &= 2K^{\lambda\nu\mu} \end{aligned}$$

$$\longrightarrow \boxed{K^{\lambda\nu\mu} = \frac{1}{2} \left(S^{\lambda\nu\mu} + S^{\nu\mu\lambda} - S^{\mu\lambda\nu} \right)}$$

We now write the explicit expression:

$$K^{\lambda\nu\mu} = \frac{(-i)}{2} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi^A)} (J_s^{\mu\nu})^A{}_B \, \phi^B + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^A)} (J_s^{\mu\lambda})^A{}_B \, \phi^B - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} (J_s^{\lambda\nu})^A{}_B \, \phi^B \right]$$

so we find the Belinfante - Rosenfeld energy - momentum tensor:

$$\boxed{\textcircled{H}^{\mu\nu} = T^{\mu\nu} - \frac{i}{2} \partial_\lambda \left[\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi^A)} (J_s^{\mu\nu})^A{}_B \, \phi^B + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^A)} (J_s^{\mu\lambda})^A{}_B \, \phi^B - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} (J_s^{\lambda\nu})^A{}_B \, \phi^B \right]}$$

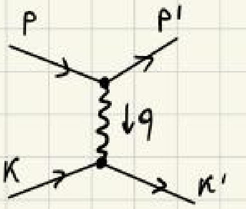
We can, therefore, do better: what we find is that a massless spin-2 particle interacts according to:

$$\boxed{\mathcal{L}_{int} = \kappa \, h_{\mu\nu}(x) \, \textcircled{H}^{\mu\nu}(x)}$$

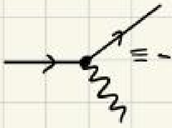
ESTIMATING THE FORCE FROM PARTICLE EXCHANGE

- QED case propagator

Consider a process of the Kind:



with



$\equiv -ieq_k \gamma^\mu$

The amplitude is:

$$i\mathcal{M} = \left[\bar{u}(p')(-ieq_k\gamma^\mu)u(p)\right] \underbrace{D_{F,\mu\nu}(q)}_{\text{photon propagator}} \left[\bar{u}(k')(-ieq_j\gamma^\nu)u(k)\right]$$

Let's try to make few general considerations about this propagator. We constructed it before explicitely using Gauge fixing, Green's functions,... Let's try to extract some fundamental property that we can hope to apply to the case of gravity.

Given our analysis in the massive spin-1 case, we expect that:

$$\underset{\mu}{\bullet}\overset{\vec{p}}{\rightsquigarrow}\underset{\nu}{\bullet} \equiv \frac{i}{p^2+i\varepsilon}\sum_{\lambda=\pm1}\varepsilon^\mu(\vec{p},\lambda)\varepsilon^\nu(\vec{p},\lambda)^*$$

is it true also in the QED case? We check. We actually already computed the sum over transverse polarizations, although in the case of $\vec{p}=|\vec{p}|\hat{z}$ we found:

$$\sum_{\lambda=\pm1}\varepsilon^\mu(|\vec{p}|\hat{z},\lambda)^*\varepsilon^\nu(|\vec{p}|\hat{z},\lambda) = \begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{pmatrix}$$

It seems to not match with the form of the propagator. However we can write the above matrix as:

$$\sum_{\lambda=\pm1}\varepsilon^\mu(|\vec{p}|\hat{z},\lambda)^*\varepsilon^\nu(|\vec{p}|\hat{z},\lambda) = -g^{\mu\nu}+\frac{p^\mu\bar{p}^\nu+p^\nu\bar{p}^\mu}{p\cdot\bar{p}} \quad \text{with} \quad p^\mu=(|\vec{p}|,0,0,|\vec{p}|)\,;\ \bar{p}^\mu=(-|\vec{p}|,0,0,|\vec{p}|)$$

Comment: Although we derived the above expression in the special case of $\vec{p}=|\vec{p}|\hat{z}$ it is actually true in general. It is indeed possible to check that for generic p^μ it remains true if we consider the transverse polarizations:

$$\varepsilon^\mu(\vec{p},\lambda=\pm1)=\frac{e^{\pm i\beta}}{\sqrt{2}}\begin{pmatrix} 0\\ iS_\rho \mp C_\rho C_\theta\\ -iC_\rho \mp S_\rho C_\theta\\ \pm S_\theta \end{pmatrix} \quad \text{with} \quad p^\mu=\begin{pmatrix} |\vec{p}|\\ |\vec{p}|C_\rho S_\theta\\ |\vec{p}|S_\rho S_\theta\\ |\vec{p}|C_\theta \end{pmatrix}$$

If we then write:

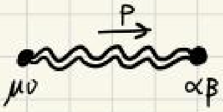
$$\frac{i}{p^2+i\varepsilon}\sum_{\lambda=\pm1}\varepsilon^\mu(\vec{p},\lambda)^*\varepsilon^\nu(\vec{p},\lambda) = \frac{i}{p^2+i\varepsilon}\left(-g^{\mu\nu}+\frac{p^\mu\bar{p}^\nu+p^\nu\bar{p}^\mu}{p\cdot\bar{p}}\right)$$

→ we notice that these terms do not contribute and can be neglected

$$\longrightarrow \boxed{\overset{\mu}{\bullet}\overset{p}{\rightsquigarrow}\overset{\nu}{\bullet} = \frac{i}{p^2+i\varepsilon}\sum_{\lambda=\pm1}\varepsilon^\mu(\vec{p},\lambda)^*\varepsilon^\nu(\vec{p},\lambda) \approx \frac{-g^{\mu\nu}}{p^2+i\varepsilon}}$$

- GRAVITY CASE propagator

We now apply the same logic to the gravity case. We try to compute the propagator as follows:



$$\boxed{= \frac{i}{p^2+i\varepsilon} \sum_{\lambda=\pm 2} \varepsilon^{\mu\nu}(\vec{p},\lambda)^* \, \varepsilon^{\alpha\beta}(\vec{p},\lambda)}$$

exploiting the fact that $\varepsilon^{\mu\nu}(\vec{p},\pm 2) = \varepsilon^{\mu}(\vec{p},\pm 1)\,\varepsilon^{\nu}(\vec{p},\pm 1)$. We consider the case $\vec{p} = |\vec{p}|\hat{z}$. As we've learned from the previous example this is not a restriction since the terms proportional to p^μ and p^ν do not contribute to the physical propagation. We have:

$$\varepsilon^{\mu\nu}(|\vec{p}|\hat{z}, \pm 2) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

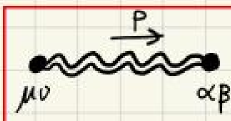
even without doing any computation, we can try to guess

$$\sum_{\lambda=\pm 2} \varepsilon^{\mu\nu}(\vec{p},\lambda)^* \, \varepsilon^{\alpha\beta}(\vec{p},\lambda) = ?$$

we look for a structure made out of $g^{\mu\nu}$ with 4 indices that must be symmetric in the exchange $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$ and that must also be traceless, separately both in $\mu\nu$ and $\alpha\beta$ when restricted to the central 2 dimensional block.
It is easy to see that one ends up with the combination:

$$\frac{1}{2}\underbrace{\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}\right)}_{\substack{\text{symmetric in the exchange } \mu\leftrightarrow\nu \\ \text{and } \alpha\leftrightarrow\beta \text{ and the relative minus} \\ \text{is there to ensure that the 1-2 block} \\ \text{is traceless both in } \mu\nu \text{ and } \alpha\beta}}$$

In conclusion we write:



$$= \frac{i}{p^2+i\varepsilon} \sum_{\lambda=\pm 2} \varepsilon^{\mu\nu}(\vec{p},\lambda)^* \, \varepsilon^{\alpha\beta}(\vec{p},\lambda) = \frac{i}{p^2+i\varepsilon}\,\frac{1}{2}\Big(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta} + \underbrace{P_{\text{dependent terms}}}_{\text{do not contribut}}\Big)$$

Exercise

Consider a real scalar massive field described by the Lagrangian: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$.
Write the energy momentum tensor $T^{\mu\nu}$ and construct the coupling $h^{\mu\nu}T_{\mu\nu}\,k = \mathcal{L}_{int}$. Write the Feynman rule and the soft limit of the emission vertex

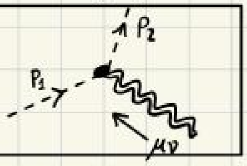
- $$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}(\partial^\nu\phi^A) - g^{\mu\nu}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\left[\frac{1}{2}(\partial_\rho\phi)(\partial^\rho\phi) - \frac{1}{2}m^2\phi^2\right] =$$
$$= (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}g^{\mu\nu}(\partial_\rho\phi)(\partial^\rho\phi) + \frac{1}{2}g^{\mu\nu}m^2\phi^2$$

we thus have:

$$\mathcal{L}_{int} = k\,h_{\mu\nu}T^{\mu\nu} = k\,h_{\mu\nu}\left[(\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}g^{\mu\nu}(\partial_\rho\phi)(\partial^\rho\phi) + \frac{1}{2}g^{\mu\nu}m^2\phi^2\right]$$

To extract the Feynman rule we consider a scattering process of the kind:

$$\boxed{S_{fi} = \langle \phi(P_2) | (-i) \int d^4x \, (-\mathcal{L}_{int}) \, | \phi(P_1) \, h^{\mu\nu}(P) \rangle}$$



There are 3 terms in this interaction:

$$S_{fi} = +i \int d^4x \, h_{\mu\nu}(x) \Big[\underbrace{(\partial^\mu\phi)(\partial^\nu\phi)}_{(ii)} - \underbrace{\tfrac{1}{2} g^{\mu\rho}(\partial_\rho\phi)(\partial^\rho\phi)}_{(iii)} + \underbrace{\tfrac{1}{2} m^2 g^{\mu\nu}\phi^2}_{(i)} \Big]$$

where:

$$\begin{cases} \phi(x) = \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{E_p}} \left(a(\vec{p}) e^{-ip\cdot x} + a^+(\vec{p}) e^{ip\cdot x} \right) = \phi_+(x) + \phi_-(x) \\ \partial_\mu \phi(x) = \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left(a(\vec{p})(-iP_\mu) e^{-ip\cdot x} + a^+(\vec{p})(iP_\mu) e^{ip\cdot x} \right) = \partial_\mu\phi_+ + \partial_\mu\phi_- \\ h^{\mu\nu}(x) = h^{\mu\nu}_+(x) + h^{\mu\nu}_-(x) \end{cases}$$

- $\phi(x) | \phi(K) \rangle = \phi_+(x) | \phi(K) \rangle = \displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} a(\vec{p}) e^{-ip\cdot x} \sqrt{2E_K} \, a^+(\vec{K}) |0\rangle = e^{-iK\cdot x} |0\rangle$
- $\partial_\mu \phi(x) | \phi(K) \rangle = (-iK_\mu) e^{-iK\cdot x} |0\rangle$
- $h^{\mu\nu}(x) | g(K) \rangle = \varepsilon^{\mu\nu}(K) e^{-iK\cdot x} |0\rangle$
- $\langle \phi(K) | \phi(x) = \langle 0 | e^{iK\cdot x}$
- $\langle \phi(K) | \partial_\mu \phi(x) = \langle 0 | (iK_\mu) e^{iK\cdot x}$

Consider interaction of type (i):

$$\frac{iKm^2}{2} \int d^4x \, \langle \phi(P_2) | h_{\mu\nu} g^{\mu\nu} \phi \phi | \phi(P_1) g(K) \rangle = \frac{iKm^2}{2} \cdot 2 \int d^4x \, e^{-ix(P_1+K-P_2)} \xi_{\mu\nu}(K) = iKm^2 g^{\mu\nu} \xi_{\mu\nu}(K) \cdot (2\pi)^4 \delta(P_1+K-P_2)$$

Consider interaction of type (ii):

$$iK \int d^4x \, \langle \phi(P_2) | h_{\mu\nu}(x) (\partial^\mu\phi)(\partial^\nu\phi) | \phi(P_1) g(K) \rangle = iK \int d^4x \, e^{-ix(P_1+K-P_2)} \varepsilon_{\mu\nu}(K) \left[(-iP_1^\nu)(iP_2^\mu) + (-iP_1^\mu)(iP_2^\nu) \right]$$

$$= iK \int d^4x \, e^{-ix(P_1+K-P_2)} \varepsilon_{\mu\nu}(K) \left[(-iP_1^\nu)(iP_2^\mu) + (-iP_1^\mu)(iP_2^\nu) \right] =$$

$$= iK \, \varepsilon_{\mu\nu}(K) (P_1^\nu P_2^\mu + P_1^\mu P_2^\nu) (2\pi)^4 \delta(P_1+K-P_2)$$

Consider interaction of type (iii):

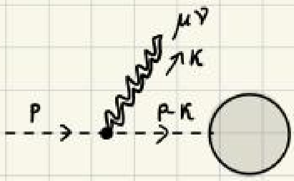
$$iK \left(-\frac{1}{2} g^{\mu\nu}\right) \int d^4x \, \langle \phi(P_2) | h_{\mu\nu} (\partial_\rho\phi)(\partial^\rho\phi) | \phi(P_1) g(K) \rangle = -\frac{iK g^{\mu\nu}}{2} \int d^4x \, e^{-ix\cdot(P_1+K-P_2)} \varepsilon_{\mu\nu}(K) \cdot 2 \cdot (-iP_1^\rho)(iP_{2,\rho}) =$$

The Feynman is then:

$$= -iK g^{\mu\nu} (P_1 \cdot P_2) \varepsilon_{\mu\nu}(K) \cdot (2\pi)^4 \delta(P_1+K-P_2)$$

$$= iK \left[g^{\mu\nu}(m^2 - P_1 \cdot P_2) + (P_1^\mu P_2^\nu + P_1^\nu P_2^\mu) \right]$$

We now consider the emission vertex and take the *soft limit*



$$\equiv \overbrace{i\mathcal{M}_0(p-K)}^{\text{without emission}} \frac{i}{(p-K)^2-m^2} \times iK \times \varepsilon^*_{\mu\nu}(K) \times \Big[\underbrace{g^{\mu\nu}(m^2-p\cdot(p-K)) + p^\mu(p-K)^\nu + p^\nu(p-K)^\mu}_{\text{using } p^2=m^2}\Big]$$

$$= i\mathcal{M}_0(p-K) \frac{i}{(p-K)^2-m^2} \times iK \times \varepsilon^*_{\mu\nu}(K) \Big[g^{\mu\nu}(\cancel{m^2}-\cancel{p^2}+p\cdot K) + p^\mu p^\nu + p^\nu p^\mu \underbrace{- p^\mu K^\nu - p^\nu K^\mu}_{\substack{\text{give } \emptyset \text{ when} \\ \text{contracted} \\ \text{with } \varepsilon_{\mu\nu}(K)}}\Big]$$

$$= i\mathcal{M}_0(p-K) \frac{i}{(p-K)^2-m^2} \times iK \times \varepsilon^*_{\mu\nu}(K) \Big[\underbrace{g^{\mu\nu}p\cdot K}_{\text{subleading}} + 2p^\mu p^\nu\Big]$$

$$= i\mathcal{M}_0(p-K) \frac{i}{(p^2+K^2-2pK-m^2)} (iK)\, \varepsilon^*_{\mu\nu}(K)\; 2p^\mu p^\nu =$$

$$\approx i\mathcal{M}_0(p)\left(\frac{(-K)}{p\cdot K} p^\mu p^\nu \varepsilon^*_{\mu\nu}(K)\right)$$

which is precisely the structure we guessed in our initial discussion with $K \equiv F(0)$.

We now come back to the derivation of the interaction mediated by the exchange of a graviton.
The idea is that in the non rel. limit, the computations performed with Feynman diagrams must introduce the results of N.R - Q.M. where the interaction between particles is described by a potential $V(\vec{x})$.

We recall the basic formulas of scattering theory in n.r. quantum mechanics. The elastic cross-section for a particle of mass m in the potential $V(\vec{x})$ has the form:

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \;;\quad f(\theta) = \frac{-m}{2\pi}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\, V(\vec{x})} \qquad \text{Born Approximation}$$

The scattering is considered to be elastic, meaning that if we denote with $\vec{p}$ the initial 3-momentum of the particle an $\vec{p}'$ the final momentum, we have:

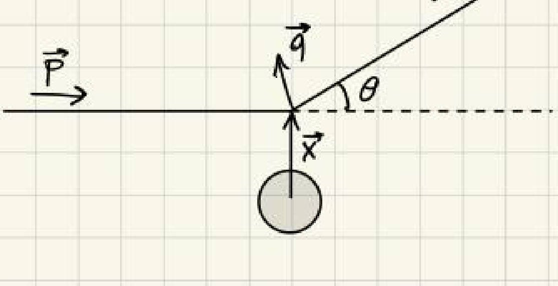
$$E_p = E_{p'} \longrightarrow \frac{|\vec{p}|^2}{2m} = \frac{|\vec{p}'|^2}{2m} \rightarrow |\vec{p}| = |\vec{p}'|$$

$\vec{q} = \vec{p}' - \vec{p}$ is the exchanged momentum and we have

$$|\vec{q}|^2 = |\vec{p}'|^2 - |\vec{p}|^2 - 2|\vec{p}|\cdot|\vec{p}'|\cdot\cos\theta = 2|\vec{p}|^2(1-\cos\theta) = 4|\vec{p}|^2\sin^2\left(\frac{\theta}{2}\right)$$

so that:

$$\boxed{\vec{q} = \vec{p} - \vec{p}'\,, \quad |\vec{q}| = 2|\vec{p}|\sin\left(\frac{\theta}{2}\right)} \qquad (\theta \text{ is the scattering angle})$$

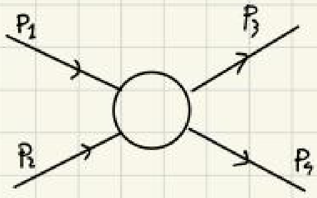


We would like to match this result with the N.R. limit of a QFT computation. Tipically there are 2 strategies

1) Match at the level of cross section.

Go back to the case of the QED. Consider a $2 \to 2$ scattering process

$$\boxed{\begin{matrix} P_1 = (E_1, \vec{P}_1) & P_3 = (E_3, \vec{P}_3) \\ P_2 = (E_2, \vec{P}_2) & P_4 = (E_4, \vec{P}_4) \end{matrix}}$$



$$\begin{cases} s = (P_1 + P_2)^2 \\ t = (P_1 - P_3)^2 \\ u = (P_1 - P_4)^2 \end{cases}$$

In the C.OM. Frame:

$$\boxed{P_1 = (E_1, \vec{P}) \qquad P_2 = (E_2, \vec{P}) \qquad P_3 = (E_3, \vec{P}') \qquad P_4 = (E_4, -\vec{P}')}$$

i) Initial state

$$\begin{aligned} P_1^2 &= E_1^2 - |\vec{P}|^2 = m_1^2 \longrightarrow |\vec{P}|^2 = E_1^2 - m_1^2 \\ P_2^2 &= E_2^2 - |\vec{P}|^2 = m_2^2 \longrightarrow |\vec{P}|^2 = E_2^2 - m_2^2 \end{aligned} \longrightarrow \boxed{E_1^2 - m_1^2 = E_2^2 - m_2^2}$$

$$s = (P_1 + P_2)^2 = (E_1 + E_2, \vec{0})^2 = (E_1 + E_2)^2 \longrightarrow \boxed{\sqrt{s} = E_1 + E_2}$$

So that:

$$\cancel{E_1^2} - m_1^2 = (\sqrt{s} - E_1)^2 - m_2^2 = s + \cancel{E_1^2} - 2\sqrt{s}\,E_1 - m_2^2 \longrightarrow \boxed{E_1 = \frac{1}{2\sqrt{s}}\left(s + m_1^2 - m_2^2\right)}$$

and in a similar fashion:

$$E_2^2 - m_2^2 = (\sqrt{s} - E_2)^2 - m_1^2 = s + E_2^2 - 2\sqrt{s}\,E_2 - m_1^2 \longrightarrow \boxed{E_2 = \frac{1}{2\sqrt{s}}\left(s + m_2^2 - m_1^2\right)}$$

and finally:

$$\begin{aligned} |\vec{P}|^2 = E_1^2 - m_1^2 &= \frac{1}{4s}\left(s + m_1^2 - m_2^2\right) - m_1^2 = \frac{1}{4s}\left(s^2 + m_1^4 + m_2^4 + 2s m_1^2 - 2s m_2^2 - 2m_1^2 m_2^2 - 4s m_1^2\right) \\ &= \frac{1}{4s}\left(s^2 + m_1^4 + m_2^4 - 2s m_1^2 - 2s m_2^2 - 2m_1^2 m_2^2\right) \equiv \frac{1}{4s}\lambda(s, m_1^2, m_2^2) \end{aligned}$$

$$\longrightarrow \boxed{|\vec{P}| = \frac{1}{2\sqrt{s}}\sqrt{\lambda(s, m_1^2, m_2^2)}} \qquad \lambda(a,b,c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2cb$$

ii) Final state

$$\begin{aligned} P_3^2 &= E_3^2 - |\vec{P}|^2 = m_3^2 \longrightarrow |\vec{P}|^2 = E_3^2 - m_3^2 \\ P_4^2 &= E_4^2 - |\vec{P}|^2 = m_4^2 \longrightarrow |\vec{P}|^2 = E_4^2 - m_4^2 \end{aligned} \longrightarrow \boxed{E_3^2 - m_3^2 = E_4^2 - m_4^2}$$

Conservation of energy gives: $\sqrt{s} = (E_1 + E_2) \overset{!}{=} (E_3 + E_4)$

$$\longrightarrow E_3^2 = E_4^2 - m_4^2 + m_3^2 = (\sqrt{s} - E_3)^2 + m_3^2 - m_4^2 = s + E_3^2 - 2\sqrt{s}\,E_3 + m_3^2 - m_4^2 \longrightarrow \boxed{E_3 = \frac{1}{2\sqrt{s}}\left(s + m_3^2 - m_4^2\right)}$$

and:

$$\longrightarrow E_4^2 = E_3^2 - m_3^2 + m_4^2 = (\sqrt{s} - E_4)^2 - m_3^2 + m_4^2 = s + E_4^2 - 2\sqrt{s}\,E_4 - m_3^2 + m_4^2 \longrightarrow \boxed{E_4 = \frac{1}{2\sqrt{s}}\left(s + m_4^2 - m_3^2\right)}$$

and:

$$\longrightarrow |\vec{p}'|^2 = \frac{1}{4s}\left(s+m_3^2-m_4^2\right)^2 - m_3^2 = \frac{1}{4s}\left(s^2+m_3^4+m_4^4+2sm_3^2-2sm_4^2-2m_3^2m_4^2-4sm_3^2\right) =$$

$$= \frac{1}{4s}\left(s^2+m_3^4+m_4^2-2sm_3^2-2sm_4^2-2m_3^2m_4^2\right) \equiv \frac{1}{4s}\lambda(s,m_3^2,m_4^2)$$

$$\longrightarrow \boxed{|\vec{p}'| = \frac{1}{2\sqrt{s}}\sqrt{\lambda(s,m_3^2,m_4^2)}}$$

The scattering cross section is given by:

$$\boxed{d\sigma = (2\pi)^4 \underbrace{\delta(P_i-P_f)}_{\text{en. mom. cons.}} \frac{1}{\underset{\text{incident flux } J\equiv\sqrt{(P_1\cdot P_2)^2-m_1^2m_2^2}}{4J}} |M_{fi}|^2 \prod_{i=fin}\frac{d^3\vec{p}_i}{(2\pi)^3\cdot 2E_i}}$$

Typically written in the form:

$$\boxed{d\sigma = \frac{1}{4J}|M_{fi}|^2 d\phi_{fin}\ ,\quad d\phi_{fin} \equiv (2\pi)^4\delta(P_i-P_f)\prod_{i=fin}\frac{d^3\vec{p}_i}{(2\pi)^3\cdot 2E_i}}$$

In our example, the final state is 2-particle. The phase space is:

$$\boxed{d\phi = (2\pi)^4\,\delta(E_1+E_2-E_3-E_4)\,\delta(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\prod_{i=3,4}\frac{d^3\vec{p}_i}{(2\pi)^3\cdot 2E_i}}$$

We evaluate it in the C.o.M. frame of the initial state $E_1+E_2=\sqrt{s}$; $\vec{p}_1+\vec{p}_2=\vec{0}$

$$d\phi = (2\pi)^4\,\delta(\sqrt{s}-E_3-E_4)\,\delta(\vec{p}_3+\vec{p}_4)\,\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\cdot\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}$$

we integrate over $\vec{p}_4$ using the delta function:

$$d\phi = (2\pi)^4\cdot\delta(\sqrt{s}-E_3-E_4)\,\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\,\frac{1}{(2\pi)^3 2E_4} = \frac{1}{(2\pi)^2}\,\frac{1}{4E_3\cdot E_4}\,\delta(\sqrt{s}-E_3-E_4)\,d^3\vec{p}_3$$

$$\longrightarrow \boxed{d\phi = \frac{1}{(2\pi)^2}\,\frac{1}{4E_3\cdot E_4}\,\delta(\sqrt{s}-E_3-E_4)\,d^3\vec{p}_3\Big|_{\vec{p}_3=-\vec{p}_4}}$$

we now write $d^3\vec{p}_3 = |\vec{p}_3|^2 d|\vec{p}_3| d\Omega$ and integrate $|\vec{p}_3|$ using the δ function

$$d\phi = \frac{1}{(2\pi)^2 4E_3E_4}\,\delta\left(\sqrt{s}-\sqrt{|\vec{p}_3|^2+|m_3|^2}-\sqrt{|\vec{p}_3|^2+|m_4|^2}\right)d\Omega\,|\vec{p}_3|^2 d|\vec{p}_3|$$

$$\longrightarrow d\phi = \frac{d\Omega}{(2\pi)^2}\int_0^\infty \frac{1}{4E_3E_4}\,\delta\left(\sqrt{s}-\sqrt{|\vec{p}_3|^2+|m_3|^2}-\sqrt{|\vec{p}_3|^2+|m_4|^2}\right)|\vec{p}_3|^2 d|\vec{p}_3|$$

now, using $\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x-x_0)$

- $\sqrt{s}-\sqrt{|\vec{p}_3|^2+m_3^2}-\sqrt{|\vec{p}_3|^2+m_4^2} \overset{!}{=} 0 \qquad \longrightarrow |\vec{p}_3| \overset{!}{=} |\vec{p}'| = \frac{1}{2\sqrt{s}}\lambda(s,m_3^2,m_4)^2$
- $f'(|\vec{p}'|) = -\frac{2|\vec{p}_3|}{2\sqrt{|\vec{p}_3|^2+m_3^2}} - \frac{2|\vec{p}_3|}{2\sqrt{|\vec{p}_3|^2+m_4^2}}\Big|_{\vec{p}_3=\vec{p}'} = -|\vec{p}'|\frac{E_3+E_4}{E_3E_4}$

$$\rightarrow d\phi = \frac{d\Omega}{(2\pi)^2}\int_0^\infty \frac{1}{4E_3E_4}\frac{E_3E_4}{|\vec{p}|\underbrace{(E_3+E_4)}_{\sqrt{s}}}\delta(|\vec{p}_3|-|\vec{p}'|)|\vec{p}_3|^2 d|\vec{p}_3| = \frac{d\Omega}{(2\pi)^2}\frac{|\vec{p}'|^2}{4\sqrt{s}|\vec{p}'|} \longrightarrow \boxed{d\phi = \frac{1}{(2\pi)^2}\frac{|\vec{p}'|}{4\sqrt{s}}d\Omega}$$

and our expression for the cross-section becomes:

$$\boxed{d\sigma = \frac{1}{4\mathcal{J}}|\mathcal{M}_{fi}|^2\frac{1}{(2\pi)^2}\frac{|\vec{p}'|}{4\sqrt{s}}d\Omega}$$

We write for the flux in the CoM frame:

$$\mathcal{J} = \left((p_1\cdot p_2)^2 - m_1^2m_2^2\right)^{1/2} \qquad p_1\cdot p_2 = (E_1,\vec{p})\cdot(E_2,-\vec{p}) = E_1E_2 + |\vec{p}|^2$$

$$s = (E_1+E_2)^2 \longrightarrow s = E_1^2 + E_2^2 + 2E_1\cdot E_2 = |\vec{p}|^2 + m_1^2 + |\vec{p}|^2 + m_2^2 + 2E_1E_2 = 2|\vec{p}|^2 + m_1^2 + m_2^2 + 2E_1E_2$$

$$s = (p_1+p_2)^2 = m_1^2 + m_2^2 + 2p_1\cdot p_2 \longrightarrow p_1\cdot p_2 = \frac{s}{2} - \frac{m_1^2}{2} - \frac{m_2^2}{2}$$

$$\mathcal{J} = \left(\frac{s^2}{4} + \frac{m_1^4}{4} + \frac{m_2^4}{4} - \frac{s}{2}m_1^2 - \frac{sm_2^2}{2} + \frac{m_1^2m_2^2}{2} - \frac{m_1^2m_2^2}{2}\right)^{1/2} = \frac{1}{2}\lambda(s, m_1^2, m_2^2)^{1/2} = \sqrt{s}|\vec{p}|$$

$$\longrightarrow \boxed{\mathcal{J} = \sqrt{s}|\vec{p}|}$$

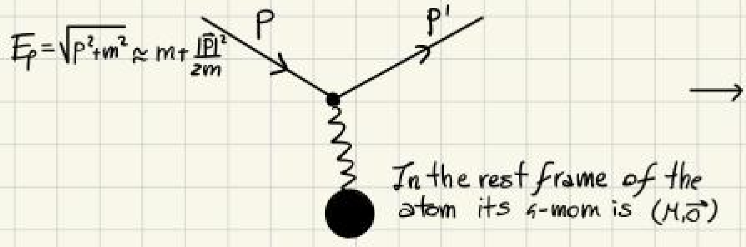
We thus arrive:

$$\boxed{d\sigma = \frac{1}{64\pi^2 s}|\mathcal{M}_{fi}|^2\frac{|\vec{p}'|}{|\vec{p}|}d\Omega}$$

We can now compare with the Born approximation. We need to make these considerations first:

1) We set $|\vec{p}| = |\vec{p}'|$: elastic scattering $\longrightarrow \boxed{d\sigma = \frac{1}{64\pi^2 s}|\mathcal{M}_{fi}|^2 d\Omega}$

2) We consider the scattering of a fermion with mass m and mom. $\vec{p}$ that scatters off a very heavy target with mass M. (non. rel limit, $p^2 \ll m \ll M$)



$$\longrightarrow s = (p_1+p_2)^2 \approx M^2 \qquad \longrightarrow \boxed{d\sigma \approx \frac{1}{64\pi^2M^2}|M|^2 d\Omega}$$

3) The computation of $d\sigma$ is based on momentum eigenstates which are normalized in a relativistic way while the Born approximation makes use of N.R. normalized mom. eigenstat

$$\begin{cases} \langle\vec{p}|\vec{q}\rangle = (2\pi)^3 2E_p\,\delta(\vec{p}-\vec{q}) & \text{relativistic normalization} \\ \langle\vec{p}|\vec{q}\rangle = (2\pi)^3\,\delta(\vec{p}-\vec{q}) & \text{non relativistic normalization} \end{cases}$$

pragmatically, we write the relation $\boxed{|\vec{p}\rangle_R = \sqrt{2E_p}|\vec{p}\rangle_{NR}}$, so that, schematically (for a 2-2 scatt.)

$$\langle f|M|i\rangle_R = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\sqrt{2E_{p_3}}\sqrt{2E_{p_4}}\langle f|M|i\rangle_{NR} \qquad \mathcal{M}_{fi}^{NR} = \frac{1}{\sqrt{2E_{p_1}2E_{p_2}2E_{p_3}2E_{p_4}}}\mathcal{M}_{fi}^R$$

in the N.R. approximation we write $(2m)(2M)$ for the multiplicative factor $\mathcal{M}_{fi}^R = (2m)(2M)\mathcal{M}_{fi}^{NR}$

$$\longrightarrow \frac{d\sigma}{d\Omega} \approx \frac{1}{\underset{4}{\cancel{64}}\pi^2 M^2}(4m^2)(4M^2)\,|\mathcal{M}_{fi}^{NR}|^2 = \frac{m^2}{4\pi^2}|\mathcal{M}_{fi}^{NR}|^2 \qquad \longrightarrow \quad \frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2}|\mathcal{M}_{fi}^{N.R.}|^2$$

we now compare the 2 cross sections and write:

$$\boxed{\frac{m}{2\pi}\mathcal{M}_{fi}^{N.R.} = f(\theta) = -\frac{m}{2\pi}\int d^3\vec{x}\, e^{-i\vec{q}\cdot\vec{x}}V(\vec{x})} \longrightarrow \boxed{V(\vec{x}) = -\int \frac{d^3\vec{q}}{(2\pi)^3}\mathcal{M}_{fi}^{N.R.}(\vec{q})\, e^{i\vec{q}\cdot\vec{x}}}$$

Comment: this procedure, at the level of the cross section, doesn't capture the relative phase of $f(\theta)$ and $\mathcal{M}_{fi}$ (since we compare essentially $|f(\theta)|^2$ and $|\mathcal{M}_{fi}|^2$). The relative phase can be computed by working directly at the level of the amplitude.

2) Match at the level of the amplitude

Consider the scattering amplitude for an eigenstate of mom. $\vec{p}$ off a static potential. We use the interaction picture in Q.M.

$$A_e = <\vec{p}'|\mathcal{U}_I(\infty,-\infty)|\vec{p}> \approx <\vec{p}'|\left[\mathbb{1} - i\int_{-\infty}^{+\infty} dt\, \hat{V}_{int}^{(I)}(t)\right]|\vec{p}> = -i\int_{-\infty}^{+\infty} <\vec{p}'|\underset{\substack{\downarrow \\ V_{int}^{(I)}(t) = e^{iH_0t}\hat{V}_{int}(\vec{x})e^{-iH_0t}}}{V_{int}^{(I)}(t)}|\vec{p}>dt = -i\int_{-\infty}^{+\infty} dt\, e^{i(E_{p'}-E_p)t}<\vec{p}'|\hat{V}_{int}(\vec{x})|\vec{p}>$$

$$= (-i)(2\pi)\,\delta(E_{p'}-E_p)<\vec{p}'|V_{int}(\vec{x})|\vec{p}>$$

So that:

$$\boxed{A_e = (-i)(2\pi)\,\delta(E_{p'}-E_p)<\vec{p}'|\hat{V}_{int}(\vec{x})|\vec{p}>}$$

We also rewrite:

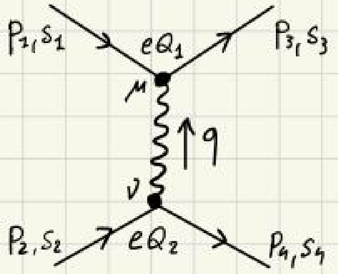
$$<\vec{p}'|\hat{V}_{int}(\vec{x})|\vec{p}> = \int d^3\vec{x}\, d^3\vec{x}'\,\underbrace{<\vec{p}'|\vec{x}'>}_{e^{-ip'\cdot x'}}\underbrace{<\vec{x}'|\hat{V}_{int}(\vec{x})|\vec{x}>}_{V_{int}(\vec{x})\delta(\vec{x}'-\vec{x})}\underbrace{<\vec{x}|\vec{p}>}_{e^{ip\cdot x}} = \int d^3\vec{x}\, e^{i\vec{x}\cdot(\vec{p}-\vec{p}')}V_{int}(\vec{x}) =$$

$$= \int d^3\vec{x}\, e^{-i\vec{x}\cdot(\vec{p}'-\vec{p})}V_{int}(\vec{x}) = \int d^3\vec{x}\, e^{-i\vec{x}\cdot\vec{q}}V_{int}(\vec{x})$$

with a Fourier transform:

$$\longrightarrow \boxed{V(\vec{x}) = \int\frac{d^3\vec{q}}{(2\pi)^3}e^{i\vec{x}\cdot\vec{q}}<\vec{p}'|\hat{V}_{int}(\vec{x})|\vec{p}>}$$

We now apply our formula to the QED case. We consider the computation of the amplitude. The idea is to consider the particle with momenta p_1 and p_3 as the scattering state in the Q.M. picture and to view the other particle as a fixed target.



$$i\mathcal{M} = \left[\bar{u}(p_3,s_3)(-ieQ_1\gamma^\mu)(u(p_1,s_1)\right]\frac{(-ig_{\mu\nu})}{q^2}\left[\bar{u}(p_4,s_4)(-ieQ_2\gamma^\nu)u(p_2,s_2)\right] =$$

$$= \frac{+ie^2Q_1Q_2}{q^2}g_{\mu\nu}\left[\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)\right]\left[\bar{u}(p_4,s_4)\gamma^\nu u(p_2,s_2)\right]$$

Consider the N.R. limit in which:

$$\begin{cases} p_1^\mu = (m,\vec{p}), & p_3^\mu = (m,\vec{p}'), \quad q^\mu = (0,\vec{p}'-\vec{p}) \longrightarrow q^2 = -|\vec{p}'-\vec{p}|^2 = -|\vec{q}|^2 \\ p_2^\mu = (M,\vec{0}), & p_4 = (M,-\vec{q}) \end{cases}$$

The Dirac spinors in the NR limit are (in the Weyl basis for γ^μ):

$$u(p,s) = \sqrt{m}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} + O\left(\frac{|\vec{p}|^2}{m}\right) \qquad \xi_1 = \begin{pmatrix}1\\0\end{pmatrix},\ \xi_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

we then find:

$$\left.\begin{aligned}
&\bullet\ \bar{u}(k,s)\gamma^0 u(p,r) = m\,(\xi_s^\dagger\ \xi_s^\dagger)\begin{pmatrix}\xi_r\\ \xi_r\end{pmatrix} = m\,(\xi_s^\dagger\xi_r + \xi_s^\dagger\xi_r) = 2m\,\delta_{sr}\\
&\bullet\ \bar{u}(k,s)\vec{\gamma}\,u(p,r) = u^\dagger(k,s)\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}u(p,r) = u^\dagger(k,s)\begin{pmatrix}-\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}u(p,s) = 0
\end{aligned}\right\}\ \text{at the first N.R. order}$$

we find:

$$i\mathcal{M} = \frac{ie^2Q_1Q_2}{-|\vec{q}|^2}(2m\,\delta_{s_1s_3})\,g_{00}\,(2M\,\delta_{s_2s_4}) \longrightarrow \boxed{\mathcal{M}^{NR} = \frac{-e^2Q_1Q_2}{|\vec{q}|^2}\delta_{s_1s_3}\delta_{s_2s_4}}$$

Therefore:

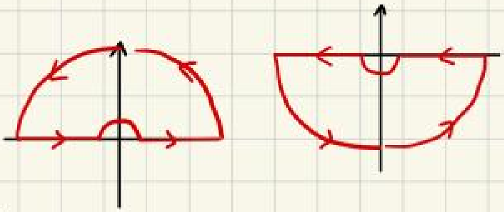
$$V(\vec{x}) = e^2Q_1Q_2\int\frac{|\vec{q}|^2 d|\vec{q}|\,2\pi\,d(\cos\theta)}{(2\pi)^3}\,\frac{e^{i|\vec{q}||\vec{x}|\cos\theta}}{|\vec{q}|^2} =$$

$$= \frac{e^2Q_1Q_2}{(2\pi)^2}\int_0^\infty d|\vec{q}|\int_{-1}^{1}d(\cos\theta)\,e^{i|\vec{q}||\vec{x}|\cos\theta} =$$

$$= \frac{e^2Q_1Q_2}{(2\pi)^2}\int_0^\infty d|\vec{q}|\,\frac{1}{i|\vec{q}|\cdot|\vec{x}|}\left(e^{i|\vec{q}|\cdot|\vec{x}|} - e^{-i|\vec{q}|\cdot|\vec{x}|}\right) = \qquad |\vec{q}| \equiv q\,,\ |\vec{x}| \equiv r$$

$$= \frac{e^2Q_1Q_2}{(2\pi)^2 ir}\int_0^\infty dq\left(\frac{e^{iqr} - e^{-iqr}}{q}\right) =$$

$$= \frac{e^2Q_1Q_2}{2(2\pi)^2 ir}\int_{-\infty}^{+\infty}dq\left(\frac{e^{iqr} - e^{-iqr}}{q}\right) =$$



$$= \frac{e^2Q_1Q_2}{8\pi^2 ir}\left[\int_{-\infty}^{+\infty}dq\,\frac{e^{iqr}}{q} - \int_{-\infty}^{+\infty}dq\,\frac{e^{-iqr}}{q}\right]$$

$$= \frac{e^2Q_1Q_2}{8\pi^2 ir}\left[\pi i\,\underset{z=0}{\mathrm{Res}}\,\frac{e^{irz}}{z} - (-\pi i)\,\underset{z=0}{\mathrm{Res}}\,\frac{e^{-irz}}{z}\right] =$$

$$= \frac{e^2Q_1Q_2}{8\pi^2 ir}(2\pi i) = \frac{e^2Q_1Q_2}{4\pi r}$$

$$\longrightarrow \boxed{V(r) = \frac{e^2Q_1Q_2}{4\pi r}}$$

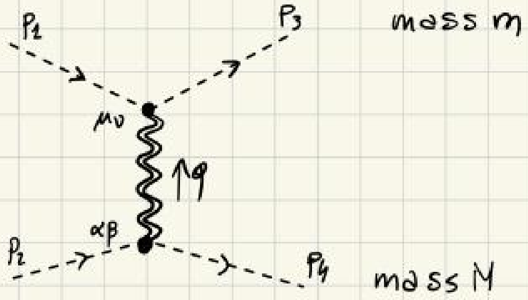
We find the Coulomb potential! Like charges repel while opposite charges attract

So we learn that the massless nature of the photon is responsible for the long range nature of electromagnetism.

Comment: if we give a small mass to the photon we would get a modification in the above computation: we get a "screening effect".

$$\boxed{V(r) = \frac{e^2Q_1Q_2}{4\pi r}e^{-m_\gamma r}}$$

We now consider a gravitational scattering amplitude



$$i\mathcal{M} = i\kappa\left[g^{\mu\nu}(m^2 - p_1\cdot p_3) + (p_1^\mu p_3^\nu + p_1^\nu p_3^\mu)\right]\frac{i}{q^2}\frac{1}{2}\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\mu\alpha} + g^{\mu\nu}g^{\alpha\beta}\right) i\kappa\left[g^{\alpha\beta}(M^2 - p_2\cdot p_4) + (p_2^\alpha p_4^\beta + p_2^\beta p_4^\alpha)\right]$$

In the NR limit, we consider:

$$p_1 = (m, \vec{p}\,), \quad p_3 = (m, \vec{p}\,') \qquad p_2 = (M, \vec{0}) \qquad p_4 = (M, -\vec{q})$$

$$p_1 \cdot p_3 = m^2 - \vec{p}\cdot\vec{p}\,' \approx m^2 \qquad p_2 \cdot p_4 \approx M^2$$

The leading term comes from $\mu = \nu = \alpha = \beta = 0$

$$\longrightarrow i\mathcal{M} = \frac{i\kappa^2}{|\vec{q}|^2}(2m^2)\,\frac{1}{2}\left(g^{00}g^{00} + g^{00}g^{00} - g^{00}g^{00}\right)(2M^2) \longrightarrow \boxed{\mathcal{M}^{NR} = \frac{\kappa^2 m M}{2|\vec{q}|^2}}$$

Finally we compute the potential:

$$V(\vec{x}) = -\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{\kappa^2 m M}{2|\vec{q}|^2}e^{i\vec{q}\cdot\vec{x}} = -\frac{\kappa^2 m M}{2(2\pi)^3}\underbrace{\int d^3\vec{q}\,\frac{e^{i\vec{q}\cdot\vec{x}}}{|\vec{q}|^2}}_{= 1/4\pi r} = -\frac{\kappa^2 m M}{8\pi r}$$

So we find the gravitational potential. From the comparison with $V(r) = \frac{-G_N m M}{r}$ we identify $\frac{\kappa^2}{8\pi} = G_N$:

$$\longrightarrow \boxed{\kappa = \sqrt{8\pi G_N}}$$

We see that gravity is always attractive.

ELEMENTS OF RENORMALIZATION

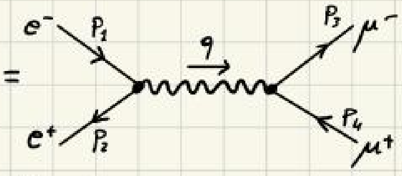
INTRODUCTION

Consider the process

$$\boxed{e^-(p_1)\,e^+(p_2) \longrightarrow \mu^-(p_3)\,\mu^+(p_4)}$$

If we compute the scattering amplitude we have that, at the tree level, only one diagram:

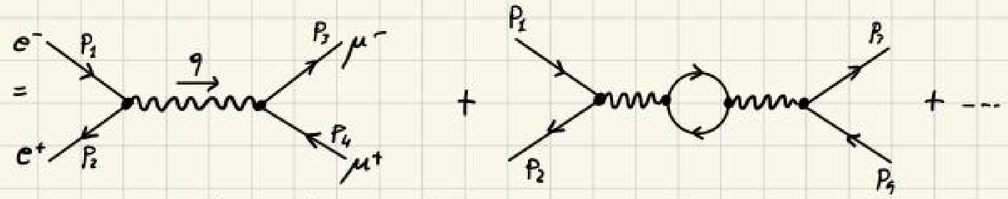
$i\mathcal{M} =$



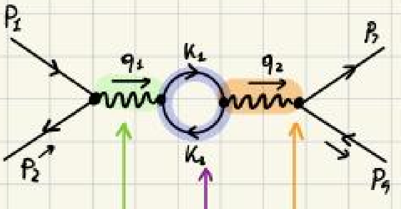
(Internal momentum fixed by conservation in terms of external momenta)

However, as we increase the order in the Dyson expansion of the scattering matrix, we encounter more complicated diagrams:

$i\mathcal{M} =$



The 2nd diagram is the so called "one loop" diagram. The peculiarity is that not all the internal momenta are fixed by conservation in terms of external legs.



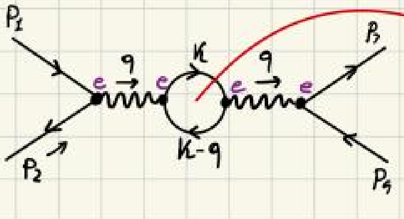
$q_1 = p_1 + p_2$

$q_2 = p_3 + p_4$ but since $p_1 + p_2 = p_3 + p_4$

$q_2 = q_1 = q$

$k_2 + q = k_1 \longrightarrow k_2 = k_1 - q = k_1 - p_1 - p_2$

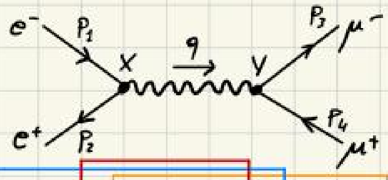
As we can se k_1 is not fixed and we have to integrate over it. For simplicity $k_1 \equiv k$.



It involves an integral

$$\int \frac{d^4k}{(2\pi)^4}$$

Naively, the tree level diagram gives a contribution to the amplitude that is proportional to e^2 while the loop diagram gives a contribution that is of higher order in a formal power series expansion in terms of powers of the electric charge "e". For example the above diagram (1-loop) is proportional to e^4, so we expect it to be subleading. Let's try to be more precise. Consider a tree-level scattering:



$$S_{fi} = \langle \mu^-(p_3)\,\mu^+(p_4)|\int d^4x\, d^4y\, (\bar{\psi}_e \gamma^\mu A_\mu \psi_e)_x (\bar{\psi}_\mu \gamma^\mu A_\mu \psi_\mu)_y\, |e^-(p_1)\, e^+(p_2)\rangle =$$

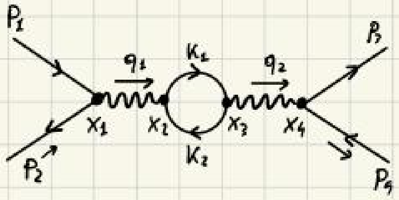
Focusing only on the x, y dependence:

$$\sim \int d^4x\, d^4y\, e^{-ixp_1} e^{-ixp_2} e^{iyp_3} e^{iyp_4} \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} =$$

$$= \int d^4x\, d^4y \int \frac{d^4q}{(2\pi)^4} e^{ix(-p_1-p_2-q)} e^{iy\cdot(p_3+p_4+q)} =$$

$$= \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \underbrace{\delta(p_1+p_2+q)\,\delta(p_3+p_4+q)}_{\substack{\text{each interaction vertex}\\ \text{produces a } \delta\text{-function}\\ \text{(one is used to fix } q\text{, the other}\\ \text{gives the tot. conservation)}}} \cdot (2\pi)^4 = (2\pi)^4 \delta(p_3+p_4-p_1-p_2)$$

Consider the one loop diagram:



$$S_{fi} = \int \prod_{i=1}^{4} d^4x_i\, e^{-ix_4\cdot(p_3+p_4)} e^{ix_1(p_1+p_2)} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \cdot e^{-iq_1(x_1-x_2)} e^{-iq_2(x_3-x_4)} e^{-ik_1(x_2-x_3)} e^{-ik_2(x_3-x_2)}$$

$$\sim \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2\pi)^4 \delta(p_1+p_2-q_1)\, (2\pi)^4 \cdot \delta(p_3+p_4-q_2)\, (2\pi)^4 \delta(q_1-k_1+k_2)\, (2\pi)^4 \delta(-q_2+k_1+k_2) =$$

$$ix_2\,(q_1-k_1+k_2$$

We fix $q_1 = p_1+p_2$, $q_2 = p_3+p_4$, $k_2 = k_1 - q_1$:

$$= \int \frac{d^4k_1}{(2\pi)^4} (2\pi)^4 \delta(-p_3-p_4\cancel{+k_1}\cancel{-k_1}+p_1+p_2) = (2\pi)^4 \delta(p_1+p_2-p_3-p_4) \int \frac{d^4k_1}{(2\pi)^4}$$

So the loop diagram, therefore, involves the integral (in the amplitude)

$$e^4 \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}}{(k^2-m^2)} \frac{\not{k}-\not{q}}{[(k-q)^2-m^2]}$$

we just count the powers of the integration variable, no attention is paid for γ functions etc.

$$\longrightarrow \sim \int d^4k\, \frac{k^2}{k^4} \sim \int \frac{d^4k}{k^2} \sim \int dk\, \frac{k^3}{k^2} = \int dk\, k \quad \text{diverges quadratically in } k$$

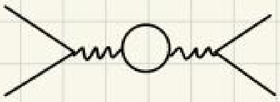
What is supposed to be a small correction actually diverges!

SUPERFICIAL DEGREE OF DIVERGENCE

We try to set the previous analysis on more formal ground. First of all, let's focus on diagrams which are connected and 1PI.

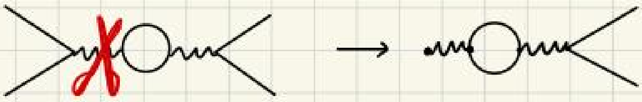
<u>CONNECTED</u> : A diagram made of external lines, vertices and propagators in which we can find a path from one element to another without jumps.

Ex:



connected!

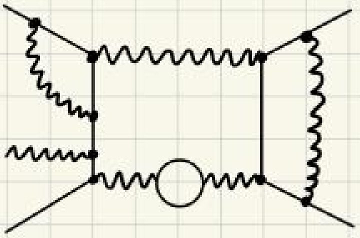
<u>1P.I.</u> : A diagram that cannot be cut in 2, by cutting a single propagator.

- 
  still connected but not 1PI
- 
  this is now 1PI (and connected.

There is no loss of generality if we limit to connected and 1PI diagrams.

- ▶ disconnected diagrams can be written as product of connected diagrams
- ▶ diagrams that are not 1PI can be obtained by gluing together 1P.I. diagrams without doing additional integrals

For example let's consider some crazy diagrams:



$$\sim \int \frac{d^4k_1 \cdots d^4k_L}{(\cancel{k}_1 - m)\, k_j^2 \cdots k_n^2}$$

"L" is the number of loops, and we integrate over the "L" momenta $k_1, \ldots, k_L$. In the integrand, the dependence on k_i will enter via propagators that give negative power of loop momenta. (N.B. This is strictly true in QED, since the vertex does not depend on momenta.)

We introduce the following definition: superficial degree of divergence.

$$\boxed{D \equiv \text{number of } k \text{ in the numerator} - \#\text{ of } k \text{ in the denominator}}$$

Given this definition the integral will have the structure:

$$\int^{\Lambda} d^4k \; k^{D-1} \sim \begin{cases} \text{if } D > 0 & \longrightarrow \text{power divergence in } \Lambda \\ \text{if } D = 0 & \longrightarrow \log \text{ divergence in } \Lambda \\ \text{if } D < 0 & \longrightarrow \text{convergent} \end{cases}$$

The task is now a more explicit computation of "D". To this end, we define

N_e	$\equiv$	# of external fermionic legs
N_γ	$\equiv$	# of external photon legs
P_e	$\equiv$	# of internal fermionic propagators
P_γ	$\equiv$	# of internal photon propagators
V	$\equiv$	# of vertices
L	$\equiv$	# of loops

We notice that:

- The fermionic propagator has the form $\frac{i(\not{K}_i + m)}{K_i^2 - m^2} \longrightarrow P_e$ contributes as $-P_e$ to D
- The photon propagator is $\frac{-i g_{\mu\nu}}{K^2 + i\varepsilon} \longrightarrow P_\gamma$ contributes as $-2P_\gamma$ to D
- There is no momentum dependence in the QED. vertex.
- Every loop integral gives a 4 powers of K

We then write:

$$\boxed{D = 4L - P_e - 2P_\gamma}$$

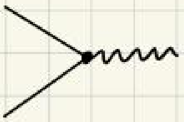
we now elaborate on this equation as follows

i) As we have discussed the number of loops in a diagram is equal to the number of internal momenta of propagators that is not fixed by momentum conservation at the vertex.

$$L = \underbrace{P_e + P_\gamma}_{\substack{\text{\# internal momenta} \\ \text{(one for each propagator)}}} - \underbrace{(V-1)}_{\substack{\text{constraints from} \\ \text{conservations at the} \\ \text{vertices.}}}$$

ii) We work on "V", the number of vertices in a diagram.

Consider an <u>individual QED vertex</u>: each one gives one photon line and two fermion lines.



Cosequently, if we have a diagram with a total of "V" vertices, we will have a total of

$$\boxed{V \text{ photon lines} + 2V \text{ fermion lines}}$$

Consider first the photon lines. As we have said the diagram has N_γ external photon lines. It means that the remaining $V - N_\gamma$ photon lines are internal and, therefore, they must be closed to form propagators. Each propagator closes two photon lines so that we have one photon propagator for every 2 internal photon lines

$$\longrightarrow \boxed{\frac{V - N_\gamma}{2} = P_\gamma}$$

Consider now fermion lines. The diagram has N_e external fermion lines. It means that $2V - N_e$ lines are internal and must be closed into propagators.
Since each propagator closes 2 fermion lines, we have:

$$\boxed{\frac{2V - N_e}{2} = P_e}$$

We now combine the equations we found so far. Simple algebra gives:

$$L = P_e + P_\gamma - (V-1) = \frac{2V - N_e}{2} + \frac{V - N_\gamma}{2} - (V-1) = V - \frac{N_e}{2} + \frac{V}{2} - \frac{N_\gamma}{2} - V + 1 \longrightarrow \boxed{L = \frac{1}{2}(V - N_e - N_\gamma) + 1}$$

We then go back to "D" and find:

$$D = 4L - P_e - 2P_\gamma = 4\,\frac{1}{2}(V - N_e - N_\gamma) + 4 - \left(\frac{2V - N_e}{2}\right) - 2\left(\frac{V - N_\gamma}{2}\right) =$$

$$= V(2-1-1) + N_e\left(-2+\frac{1}{2}\right) + N_\gamma(-2+1) + 4 =$$

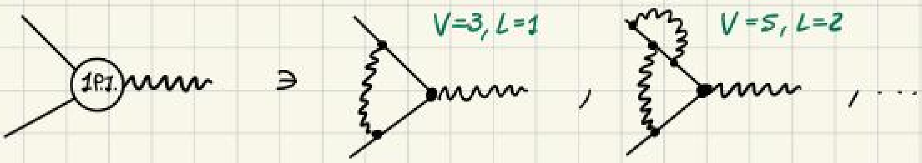
$$= 4 - N_\gamma - \frac{3}{2}N_e$$

$$\longrightarrow \boxed{D = 4 - N_\gamma - \frac{3}{2}N_e}$$ Superficial degree of divergens in QED

Comments:

We rewrote D in terms only of (V, N_γ, N_e), instead of (L, P_e, P_γ). In this process, we find that the dependence of "V" actually cancels at in the final expression so that D is only function of N_γ and N_e.
This is very relevant for the following reason. Instead of talking about the degree of divergence of an individual connected and 1PI diagram, we can talk about the degree of divergence of a connected 1PI amplitude which is only identified by the number of external fermions and photon legs, independently from its number of vertices V.

Example: suppose we want to describe the case with $N_e = 2$, $N_\gamma = 1 \longrightarrow D = 0$.



Consequently, we can classify the connected, 1PI amplitude depending on their value of D

We notice that we have a finite number of superficially divergent amplitudes but an infinite number of superficially divergent diagrams.

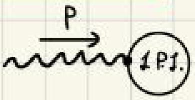
$N_\gamma \backslash N_e$	0	2	4	
0	4	1	-2	
1	3	0	-3	
2	2	-1	-4	
3	1	-2	-5	
4	0	-3	-6	

SUPERFICIAL DEGREE OF DIVERGENCE AND SYMMETRIES

"D" is defined in a very naive way. Consequently we expect it to be "bund" to structural properties of the theory like for instance, the presence of symmetries. We consider few examples

SUPERFICIALLY DIVERGENT AMPLITUDES THAT ACTUALLY VANISH AT ALL ORDERS

- Consider the case $\boxed{N_e=0,\ N_\gamma=1,\ D=3}$. We draw this amplitude as:

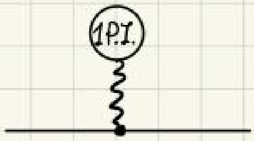


A.K.A. TADPOLE DIAGRAMS

We take this occasion to make an important comment of general validity. We write the amplitude as follows, if we consider the photon to be external.

$$i\mathcal{M} = \varepsilon_\mu(p)\, i\mathcal{M}^\mu(p)$$

However, we don't want to make any assumption about the role of this photon. For instance, in the logic we're following, it is well possible that this 1P.I. amplitude is part of a bigger non-1P.I. amplitude of the type



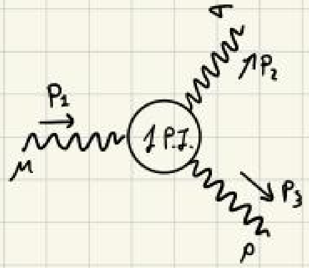
In this case the photon is part of a propagator and we have $i\mathcal{M}^\mu(p)$ attacked to a propagator $-ig_{\mu\nu}/p^2$. We'll focus always on quantity like $i\mathcal{M}^\mu(p)$ in which we strip all the information about the nature of the external legs.

Let's go back to the tadpole diagram. On very general ground, we will have:

$$i\mathcal{M}^\mu(p) = p^\mu f(p^2)$$

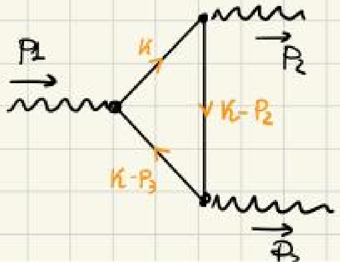
as a consequence of Lorentz covariance. However momentum conservation requires $p^\mu=0$. Consequently $\boxed{i\mathcal{M}^\mu(p)=0}$. In this case "D" is bund w.r.t. Lorentz.

- Consider the case $\boxed{N_e=0,\ N_\gamma=3,\ D=1}$. We draw the amplitude as:



$$\equiv i\mathcal{M}^{\mu\rho\sigma}(p_1,p_2,p_3)$$

We compute it at one loop. We draw the diagram:



$$\longrightarrow i\mathcal{M}^{\mu\rho\sigma} = \int \frac{d^4k}{(2\pi)^4}(ie)^3\,\mathrm{tr}\left[\underbrace{\frac{i(\not{k}+m)}{k^2-m^2}\gamma^\mu \frac{i(\not{k}-\not{p}_2-\not{p}_3+m)}{(k-p_2-p_3)^2-m^2}\gamma^\rho \frac{i(\not{k}-\not{p}_2+m)}{(k-p_2)^2-m^2}\gamma^\sigma}_{\text{we start from an arbitrary vertex or propagator and we follow backward the fermion line}}\right](-1)$$

integral over the loop mom.

for each vertex (ie)

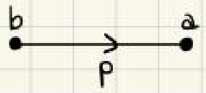
for a close fermion loop we take the trace

we add an overall (-1) for each fermion loop

Comments :

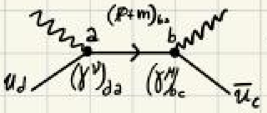
- **Why we trace?**

Remember that the fermion propagator is a matrix



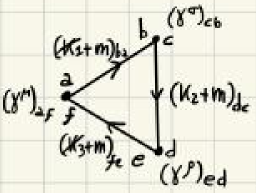
$$= \frac{i(\not p + m)_{ab}}{p^2 - m^2 + i\varepsilon}$$

In a tree-level diagram like:



these indices are eventually contracted with external spinors.

In a loop, on other hand, the propagators are contracted over themselves: so that $(\not K_1+m)_{ba}(\gamma^\mu)_{af}(\not K_3+m)_{fe}(\gamma^\rho)_{ed}(\not K_2+m)_{dc}(\gamma^\sigma)_{cb}$ and the sum closes the trace.

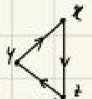


- **Why the overall (-1)?**

This is tricky. This is related to the following fact. The fermion propagator is:

$$\boxed{\langle 0|\, T[\psi(x)\bar{\psi}(y)]\,|0\rangle \equiv \overbrace{\psi(x)\bar{\psi}}(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i(\not p+m)}{p^2-m^2+i\varepsilon}}$$

In the fermionic loop we have:



$(\bar{\psi}\gamma^\mu\psi)_x(\bar{\psi}\gamma^\nu\psi)_y(\bar{\psi}\gamma^\rho\psi)_z$ we always have a contraction $\overbrace{\bar{\psi}\psi}$ instead of $\overbrace{\psi\bar{\psi}}$. This is still a propagator but with a (-1) because we need to exchange $\bar{\psi}$ and ψ.

We can elaborate more on the amplitude we've written.

$$i\mathcal{M}^{\mu\rho\sigma} = (-1)(ie)^3 \int \frac{d^4K}{(2\pi)^4}\, \frac{i^3\,\mathrm{tr}\left[(\not K+m)\gamma^\mu(\not K-\not P_2-\not P_3+m)\gamma^\rho(\not K-\not P_2+m)\gamma^\sigma\right]}{(K^2-m^2)\left[(K-P_2)^2-m^2\right]\left[(K-P_2-P_3)^2-m^2\right]} =$$

$\mathrm{tr}(\gamma^1\cdots\gamma^n)=0$ if $n=$ even $\curvearrowright$

$$= (-1)(ie)^3 i^3 \int \frac{d^4K}{(2\pi)^4}\, \frac{1}{(K^2-m^2)\left[(K-P_2)^2-m^2\right]\left[(K-P_2-P_3)^2-m^2\right]} \times \Big\{ \mathrm{tr}\left[\not K\gamma^\mu(\not K-\not P_2-\not P_3)\gamma^\rho(\not K-\not P_2)\gamma^\sigma\right] +$$

$$+ m^2\,\mathrm{tr}\left[\gamma^\mu\gamma^\rho(\not K-\not P_2)\gamma^\sigma + \gamma^\mu(\not K-\not P_2-\not P_3)\gamma^\rho\gamma^\sigma + \not K\gamma^\mu\gamma^\rho\gamma^\sigma\right]\Big\}$$

This amplitude does not vanish. However, we realize that we have a second diagram which partecipates to the amplitude:

$$i\mathcal{M}^{\mu\rho\sigma} = (ie)^3 i^3(-1)\int \frac{d^4K}{(2\pi)^4}\, \frac{1}{(K^2-m^2)\left[(K-P_2)^2-m^2\right]\left[(K-P_2-P_3)^2-m^2\right]^2} \times \mathrm{tr}\left[(-\not K+m)\gamma^\sigma(-\not K+\not P_2+m)\gamma^\rho(-\not K+\not P_2+\not P_3+m)\gamma^\mu\right]$$

Let's focus on the trace:

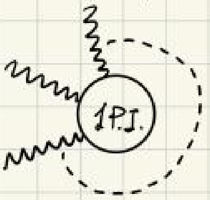
$$= \mathrm{tr}\left[-\not K\gamma^\sigma(-\not K+\not P_2)\gamma^\rho(-\not K+\not P_2+\not P_3)\gamma^\mu\right] + m^2\,\mathrm{tr}\left[\gamma^\sigma\gamma^\rho(-\not K+\not P_2+\not P_3)\gamma^\mu + \gamma^\sigma(-\not K+\not P_2)\gamma^\rho\gamma^\mu + (-\not K)\gamma^\sigma\gamma^\rho\gamma^\mu\right] =$$

we use that $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \mathrm{tr}(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu)$

$$= (-1)\,\mathrm{tr}\left[\not K\gamma^\mu(-\not K+\not P_2+\not P_3)\gamma^\rho(-\not K+\not P_2)\gamma^\sigma\right]$$

This is equal and opposite to the trace of the previous diagram. *Therefore the sum of the 2 diagrams identically vanishes.*

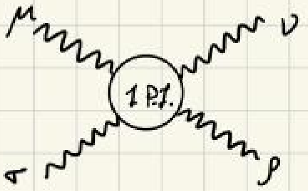
This result persists to all orders and it is called Furry's theorem. It is actually valid for all amplitude with an odd number of photonic lines. N.B. it is a consequence of invariance under charge conjugation



odd n
= 0

■ SUPERFICIALLY DIVERGENT AMPLITUDES THAT DO NOT DIVERGE

- Consider the case $\boxed{N_\gamma=4,\ N_e=0,\ D=0}$:



it should diverge (D=0) but this is actually finite.

To understand its finitness we consider the following property of "D". Consider again a generic amplitude written in the form

$$i\mathcal{M}(P_i)=\int\frac{d^4K_1\cdots d^4K_L}{K_1^2\cdots[(K_i-P_i)^2-m^2]\cdots K_n^2}$$

P_i are external momenta

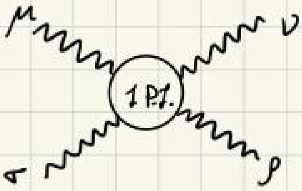
Take the derivative with respect to P_i, we get:

$$\frac{\partial\mathcal{M}}{\partial P_i^\mu}=\int\frac{d^4K_1\cdots d^4K_L}{K_1^2\cdots[(K_i-P_i)^2-m^2]^2\cdots K_n^2}\,2(K_i-P_i)_\mu$$

If we compute the degree of divergence we find: $D'=(4L-2n)-2+1=D-1$
So the degree of divergence is reduced by 1. Consequently if we Taylor expand the original amplitude we write:

$$\underset{\substack{\uparrow\\ \text{original amplitude}\\ \text{with } D}}{\mathcal{M}(P_i)}=\underset{\substack{\uparrow\\ \text{original ampl.}\\ \text{with ext. momenta}\\ \text{set to } \phi.\text{ It still}\\ \text{has } D}}{\mathcal{M}(P_i=0)}+\underbrace{\frac{\partial\mathcal{M}}{\partial P_i^\mu}\bigg|_0 P_i^\mu}_{\text{these terms have } D-1}+\underbrace{\ldots}_{\to\ \text{further derivatives lower D even more}}$$

If we come back to our example:



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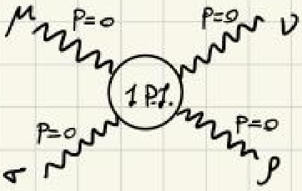


$$+\ \underbrace{---}_{D<0\ \to\ \text{they converge}}$$

In equations we write:

$$\mathcal{M}^{\mu\nu\rho\sigma}(P_1,P_2,P_3,P_4)=\underbrace{\mathcal{M}^{\mu\nu\rho\sigma}_{UV}(P_i=0)}_{\text{we isolate the divergent part}}+\mathcal{M}^{\mu\nu\rho\sigma}_{FINITE}(P_i)$$

Using Lorentz covariance we have:

$$\boxed{\mathcal{M}^{\mu\nu\rho\sigma}(p_i=0) = (g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\mathcal{M}_{UV}}$$

divergent Lorentz scalar

since $p_i=0$ we can only use $g^{\mu\nu}$ to reproduce the tensor structure

Because of Bose symmetry, the expression must be fully symmetric under a generic permutation of indices. We now use Ward identity, we write:

$$p_{1,\mu}\mathcal{M}^{\mu\nu\rho\sigma}(p_i) = p_{1,\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{UV}(p_i=0) + p_{1,\mu}\mathcal{M}^{\mu\nu\rho\sigma}_{FIN}(p_i) \overset{!}{=} 0$$

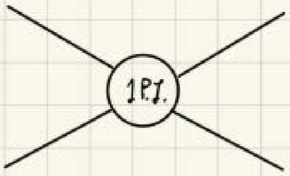
These two terms must vanish independently by construction. Consequently:

$$p_{1,\mu}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\mathcal{M}_{UV} = (p_1^{\nu}g^{\rho\sigma} + p_1^{\rho}g^{\nu\sigma} + p_1^{\sigma}g^{\nu\rho})\mathcal{M}_{UV} \overset{!}{=} 0 \longrightarrow \boxed{\mathcal{M}_{UV}=0}$$

In this case the Ward-identity forces the UV-divergent part of the amplitude to vanish. "D" in this case is "blind" to charge conservation.

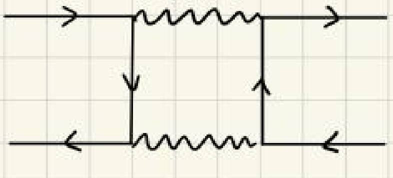
■ SUPERFICIALLY CONVERGENT AMPLITUDES THAT DIVERGE

Consider $\boxed{N_e = 4,\ N_\gamma = 0,\ D = -2}$



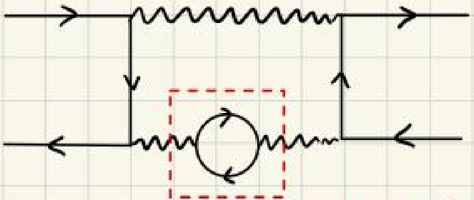
it should converge since D<0

At one-loop we have diagrams like



this diagram indeed converges

However at two-loops we have:



we still have D = -2 but it diverges!

divergent subdiagram

The point is that these are not "new" divergences. In other words, if we find a way to eliminate the divergence in the object

we automatically fix the diagram above.

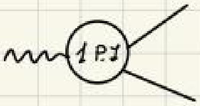
TRULY DIVERGENT DIAGRAMS (examples)

$N_e = N_\gamma = 0$; $D = 4$

$N_\gamma = 2$, $N_e = 0$; $D = 2$

$N_e = 2$, $N_\gamma = 0$; $D = 1$

$N_e = 2$, $N_\gamma = 1$; $D = 0$



D in generic Q.F.T.

Consider a QFT with an arbitrary number of scalars, fermions and vector fields.

i) Suppose that in the Lagrangian density that describes the theory we have different kind of interactions that we label with the index $i = 1, 2, \dots, n_I$.

ii) We indicate with δ_i the number of space-time derivatives that characterize the interaction "i".

Example : SCALAR QED

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - \lambda |\phi^\dagger \phi|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad ; \; D_\mu \equiv \partial_\mu \phi + ie A_\mu \phi$$

If we write explicitily the covariant derivative we find the following interactions :

1) $\lambda (\phi^\dagger \phi)^2 \longrightarrow$ no derivatives , $\delta_1 = 0$

2) $e^2 A_\mu A^\mu \phi^\dagger \phi \longrightarrow$ no derivatives , $\delta_2 = 0$

3) $ie A_\mu \left[(\partial_\mu \phi^\dagger) \phi - \phi^\dagger (\partial_\mu \phi) \right] \longrightarrow$ one derivative , $\delta_3 = 1$

So in this case we have $n_I = 3$ types of interactions with $\delta_1 = \delta_2 = 0$, $\delta_3 = 1$.

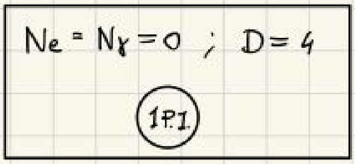
The fact that interactions do or do not have derivatives is of course crucial for our arguments. In fact, interactions with derivatives have powers of momenta in their Feynman rule. For instance :

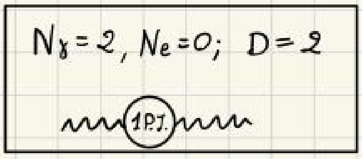
$= ie\,(P_1 + P_2)^\mu$ → so that we have power of momenta in the interaction vertex

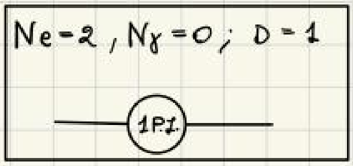
We now have an additional ingredient in our power counting. If the interaction vertex of type "i" has δ_i derivatives, it contributes with δ_i powers of momentum in the counting of D. We indicate:

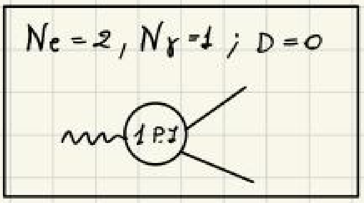
P_f : number of propagators of type "f".
N_f : number of external lines of type "f".
V_i : number of interaction vertices of type "i".

■ TRULY DIVERGENT DIAGRAMS (examples)


$N_e = N_\gamma = 0$; $D = 4$


$N_\gamma = 2$, $N_e = 0$; $D = 2$


$N_e = 2$, $N_\gamma = 0$; $D = 1$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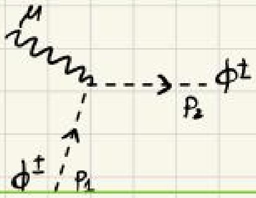

$\equiv ie\,(P_1 + P_2)^\mu$ → so that we have power of momenta in the interaction vertex

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P_f : number of propagators of type "f".
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V_i : number of interaction vertices of type "i".

We have the momentum-dependence of propagators. We opt for the following notation:

$$\boxed{\Delta_f(p) \sim p^{2S_f-2}} \qquad \text{and we have} \quad \begin{cases} S_f = 0 & \text{for scalars and photons} \\ S_f = \frac{1}{2} & \text{for fermions} \end{cases}$$

↑ asymptotic behaviour of the type "f" propagator

<u>Comments</u>: in the photon case $\Delta_\gamma(p) \sim \frac{1}{p^2}$ since we drop terms that because of Gauge invariance have no effect.
In the case of massive vectors, we know that in general we have:

$$= \frac{i}{p^2 - M^2 + i\varepsilon}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}\right) \longrightarrow \text{the asymptotic behaviour of the prop. has } S_f = 1.$$

So massive vector fields are indeed a bit tricky and we'll not consider them in the following discussion.

We now generalize the equation $D = 4L - P_e - 2P_\gamma$. We write:

$$\boxed{D = 4L + \underbrace{\sum_f (2S_f - 2)P_f}_{} + \underbrace{\sum_i \delta_i V_i}_{}}$$

# of loops given by a generalization of $L = P_e + P_\gamma - (V-1)$, that is: $L = \sum_f P_f - (\sum_i V_i - 1)$

contribution from propagators. In the QED case we sum over $f = e, \gamma$ with $S_e = \frac{1}{2}$ and $S_\gamma = 0$ to get $-P_e - 2P_\gamma$

we add a contribution from vertices with derivatives

Writing it more explicitely:

$$D = 4\left[\sum_f P_f - \left(\sum_i V_i - 1\right)\right] + \sum_f (2S_f - 2)P_f + \sum_i \delta_i V_i =$$
$$= \sum_f P_f (2S_f - 2 + 4) + \sum_i V_i (\delta_i - 4) + 4$$

$$\longrightarrow \boxed{D = \sum_f P_f (2S_f + 2) + \sum_i V_i (\delta_i - 4) + 4}$$

We now need to relate this expression with the external lines. To this end, we need to introduce a new definition. We indicate with:

$n_{i,f}$: the number of fields of type "f" present in the interaction vertex of type "i".

We try to generalize the relations: $V = 2P_f + N_\gamma$ and $V = P_e + \frac{N_e}{2}$. The logic goes as follows: consider a diagram with a total number of vertices given by $\sum_i V_i$. Consequently the total number of fields of type "f" present in the diagram is, therefore $\sum_i V_i \cdot n_{i,f}$

- If we have N_f external lines of type "f" it means that the remaining lines of type "f" $\sum_i V_i \cdot n_{i,f} - N_f$ are close into propagators

■ We thus have $\boxed{\frac{\sum_i V_i \cdot n_{i,f} - N_f}{2} = P_f}$ (one equation for each "f").

We thus have:

$$D = \sum_f \left[(2S_f+2)\frac{1}{2}\left(\sum_i V_i n_{i,f} - N_f\right)\right] + \sum_i V_i(\delta_i - 4) + 4 =$$

$$= 4 - \sum_f (S_f+1)N_f + \sum_i V_i(\delta_i - 4) + \sum_i V_i \sum_f (S_f+1) n_{i,f} =$$

$$= 4 - \sum_f (S_f+1)N_f + \sum_i V_i\left[\delta_i - 4 + \sum_f (S_f+1) n_{i,f}\right]$$

$$\longrightarrow \boxed{D = 4 - \sum_f (S_f+1)N_f - \sum_i V_i\left[4 - \delta_i - \sum_f n_{i,f}(\delta_f+1)\right]}$$

Exercise:

In the case of QED, we can check that the coefficient of V_i is equal to zero. In fact we have one interaction with $\delta = 0$, $n_e = 2$, $n_\gamma = 1$; $S_e = \frac{1}{2}$, $S_\gamma = 0$ and we get:

$$V\left[4 - 0 - 2\left(\tfrac{1}{2}+1\right) - 1(0+1)\right] = V(4-3-1) = 0$$

and the V-dependence cancels out

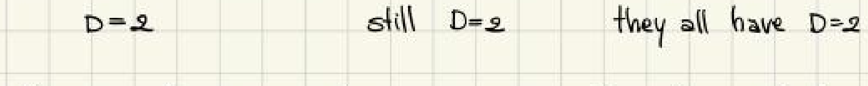
■ POWER COUNTING RENORMALIZABILITY OF A THEORY

The fact that, in general, D depends on V has important consequences. We distinguish between three cases. For simplicity consider the case in which we only have one type of interaction

$$\boxed{D = 4 - \sum_f (S_f+1)N_f - V\left[4 - \delta - \sum_f n_f(\delta_f+1)\right]}$$

i) The coefficient of V is zero and the V-dependence disappears from D (This is the QED case)
As we've discussed, in this case we have a finite number of superficially divergent amplitudes that can be classified based on N_f (but an infinite number of of superficially divergent diagrams)

ii) The dependence on V enters in D with a negative coefficient $D = 4 - \sum_f N_f(S_f+1) - V\overbrace{\left[4 - \delta - \sum_f n_f(\delta_f+1)\right]}^{\text{positive}}$
In the QED case, the dependence on V disappears and for a given superficially divergent amplitude we have an infinite number of superficially divergent diagrams since adding vertices does not change 'D'. For example:



In this case, however, if we consider a superficially divergent diagram with a fixed number of external legs N_f (say, at one loop) and we consider the same amplitude at higher orders by adding vertices, we lower the degree of superficial divergence. In this case we have a finite number of superficially divergent diagrams.

iii) The dependence on "V" enters with a positive coefficient $D = 4 - \sum_f N_f(S_f+1) - V\overbrace{\left[4 - \delta - \sum_f n_f(S_f+1)\right]}^{\text{negative}}$
This is an interesting situation. Even if we consider an amplitude with a very large number of external legs (so that $-\sum_f N_f(S_f+1)$ gives a very large negative contribution to D) if we consider such amplitude at a sufficient high order by adding vertices, we give positive contribution and at the

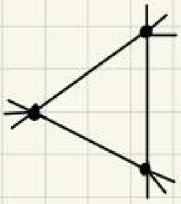
end D become negative. In this case, adding vertices rises the degree of divergence of a given amplitude. In this case we have an infinite number of superficially divergent amplitudes.

Example:

Consider $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{5!}\phi^5$. In this case $S_f = 0$; $\delta = 0$; $n_f = 5$

$$\longrightarrow D = 4 - N(0+1) - V(4-0-5(0+1)) = 4 - N + V \longrightarrow \boxed{D = 4-N+V}$$

If we consider $N = 9$ we have:



$$D = 4-9+3 = -2 < 0 \rightarrow \text{convergent!}$$

However, if we keep N fixed and consider an higher loop contribution with more V, eventually D will turn positive.

We would like to have a better understanding about what the coefficient of V is.
Let's give a better look at the vertex V:

- it has a number of derivatives equal to δ
- we have n_f fields of type "f" partecipating to it

Cosequently, it is described by an operator of the form: (at the level of the Lagrangian density)

$$\boxed{\mathcal{L}_{int}^{(V)} = \underbrace{g}_{\text{g is some coupling constant}} \partial^\delta \prod_f \phi_f^{n_f}}$$

Let us analize the mass dimension of this interaction Lagrangian. Consider the mass dimension of the field ϕ_f. We link it to the scaling of the propagator.

$$\underbrace{\langle 0 | T[\phi_f(x)\phi_f(y)] | 0 \rangle}_{\text{the mass dim. is given by } [\phi_f]^2} = \Delta_f(x-y) = \underbrace{\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \Delta_f(p)}_{\text{the mass dim. of this object is}}$$

$$\underbrace{[M]^4}_{\text{from } \int \frac{d^4p}{(2\pi)^4}} \cdot \underbrace{[\Delta_f(x)]}_{\text{from } \Delta_f(p) \sim p^{2S_f-2}} = [M]^{4+2S_f-2}$$

$$\longrightarrow [\phi_f]^2 \overset{!}{=} [M]^{4+2S_f-2} = [M]^{2+2S_f} \longrightarrow \boxed{[\phi_f] = [M]^{S_f+1}}$$

As a check:

$$\begin{cases} \text{Scalar field } (S_f=0) \rightarrow [\phi] = [M] \\ \text{fermion field } (S_f = \frac{1}{2}) \rightarrow [\psi] = [M]^{\frac{3}{2}} \\ \text{photon field } (S_f = 0) \rightarrow [A^\mu] = [M] \end{cases}$$

Since $[\mathcal{L}] = [M]^4$, we find:

$$[M]^4 \overset{!}{=} [g] \cdot [M]^{+\delta} \cdot [M]^{\sum_f n_f S_f + n_f} \longrightarrow [g] = [M]^{4-\delta-\sum_f n_f(S_f+1)} \equiv [M]^\Delta$$

$$\longrightarrow \boxed{\Delta \equiv 4 - \delta - \sum_f n_f(S_f+1)}$$

mass dimension of the coupling constant associated to the interaction vertex V

All in all:

$$\boxed{\begin{aligned} D &= 4 - \sum_f N_f (S_f + 1) + V\Delta \\ \Delta &\equiv 4 - \delta - \sum_f n_f (S_f + 1) \end{aligned}}$$

The coefficient Δ that multiplies "V" in the formula for "D" is nothing but the mass dimension of its coupling.

In Q.E.D. $\mathcal{L}_{int} = e \overline{\Psi} \gamma^\mu A_\mu \Psi$ ⟶ "e" is dimensionless and $\Delta = 0$.

We thus have the following relations:

1) Theory with a renormalizable interaction ⟶ the mass dimension of the coupling is equal to zero
Consequently, "V" does not enter in "D" and we have a theory with a finite number of superficially divergent amplitudes. Such theory is called renormalizable.

2) Theory with a non-renormalizable interaction → the mass dimension of the coupling is negative
The theory has an infinite number of superficially divergent amplitudes. Such theory is called non-renormalizable.

3) Theory with a super-renormalizable interaction ⟶ the mass dimension of the coupling is positive
The theory has a finite number of superficially divergent diagrams. Such theory is called super-renormalizable.

More in general a theory has different kind of interactions. To each one of them, we associate the dimension Δ_i of the corresponding coupling, with:

$$\boxed{D = 4 - \sum_f N_f (S_f + 1) - \sum_i V_i \Delta_i}$$

1) If $\Delta_i > 0$ for all $i = 1, \dots, n_I$ the theory is called super-renormalizable.

2) If $\Delta_i \geqslant 0$ and at least one $\Delta_i = 0$, the theory is called renormalizable.

3) If at least one $\Delta_i < 0$, the theory is called non-renormalizable.

<u>Example</u>: <u>SCALAR QED</u>

$$\boxed{\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

The interactions are:

$$\left.\begin{aligned} &\cdot\ \lambda (\phi^\dagger \phi)^2 \longrightarrow \text{renormalizable} \\ &\cdot\ e^2 A_\mu A^\mu (\phi^\dagger \phi) \longrightarrow \text{renormalizible} \\ &\cdot\ ie A_\mu \left[(\partial_\mu \phi^\dagger) \phi - \phi^\dagger (\partial_\mu \phi) \right] \longrightarrow \text{renormalizible} \end{aligned}\right\} \text{The theory is renormalizible}$$

<u>Example 2</u>: <u>MODIFIED QE.D</u>

$$\boxed{\mathcal{L} = \overline{\Psi} (i \gamma^\mu \partial_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e \overline{\Psi} \gamma^\mu A_\mu \Psi + \frac{g}{\Lambda} F_{\mu\nu} \overline{\Psi} \sigma^{\mu\nu} \psi}$$

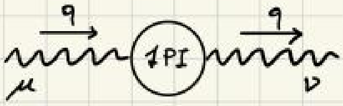
The interactions are:

$$\left.\begin{array}{l} \cdot\ e\,\overline{\psi}\gamma^\mu A_\mu \psi \longrightarrow \text{renormalizible} \\ \cdot\ \frac{g}{\Lambda} F_{\mu\nu}\overline{\psi}\sigma^{\mu\nu}\psi \longrightarrow \text{non-renormalizible} \end{array}\right\} \text{the theory is not renormalizible}$$

Example 3 : GRAVITY $\quad g = \sqrt{G_N} \sim \frac{1}{M} \rightarrow \Delta < 0 \longrightarrow$ non-renormalizable

EXPLICIT COMPUTATION AT 1-LOOP REGULARIZATION : The photon self-energy

We consider the photon 2-point amplitude



$$= i\,\Pi^{\mu\nu}(q)$$

we start from some generic considerations:

1) <u>Lorentz covariance</u>: $\Pi^{\mu\nu}(q)$ is a <u>Lorentz</u> tensor built with q^μ and $g^{\mu\nu}$. It must have a structure

$$\boxed{\Pi^{\mu\nu}(q) = \text{\textcircled{H}}(q^2)\, g^{\mu\nu} - \Pi(q^2)\, q^\mu q^\nu}$$ where $\text{\textcircled{H}}$ and Π are scalar functions

2) <u>Ward - Takahashi identity</u> : $\boxed{q_\mu \Pi^{\mu\nu}(q) = 0}$

$$\longrightarrow \text{\textcircled{H}}(q^2) q_\mu g^{\mu\nu} - \Pi(q^2)\, q_\mu q^\mu q^\nu = \text{\textcircled{H}}(q^2)\, q^\nu - \Pi(q^2)\, q^2 q^\nu = q^\nu \left[\text{\textcircled{H}}(q^2) - \Pi(q^2) q^2\right] \overset{!}{=} 0$$

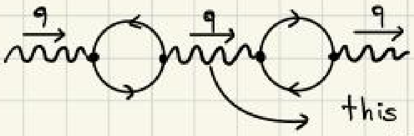
$$\longrightarrow \boxed{\text{\textcircled{H}}(q^2) = q^2\, \Pi(q^2)}$$

Therefore:

$$\boxed{\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2)}$$

3) <u>Regularity</u>: as $q^2 \to 0$ the function $\Pi(q^2)$ cannot have a pole

A pole as $q^2 \to 0$ would arise, for instance, in the case of the diagram in which we have a single photon intermediate state:



this intermediate propagator would give us a pole as $q^2 \to 0$

However this diagram is not 1PI and does not enter in 

4) <u>Degree of divergence</u>

Consider the expansion :

$$\Pi^{\mu\nu}(q) = \underbrace{\Pi^{\mu\nu}(q=0)}_{D=2} + \underbrace{\left.\frac{\partial \Pi^{\mu\nu}}{\partial q^\rho}\right|_{q=0} q^\rho}_{D=1} + \underbrace{\left.\frac{\partial \Pi^{\mu\nu}}{\partial q^\rho \partial q^\sigma}\right|_{q=0} q^\rho q^\sigma}_{D=0} + \underbrace{\dots}_{D=-1}$$

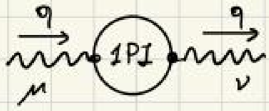
we now compare it with the previous expression:

i) if we set $q \to 0$ in $\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\, \Pi(q^2)$ we get zero (remember that $\Pi(q^2)$ is regular. $\longrightarrow$ $\Pi^{\mu\nu}(q^2=0) = 0$

ii) terms linear in q are also absent $\longrightarrow$ $\left.\frac{\partial \Pi^{\mu\nu}}{\partial q^\rho}\right|_{q=0} q^\rho = 0$

iii) the highest degree of divergence of $\Pi^{\mu\nu}(q)$, therefore <u>is D=0 and not D=2</u>

In conclusion:

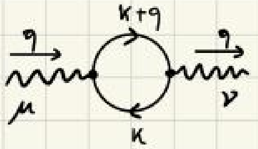


$$\boxed{\equiv i\Pi^{\mu\nu}(q) = i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)}$$

- $D=0$
- $\Pi(q^2)$ regular as $q^2 \to 0$

Explicit computation at 1 loop : perturbative renormalization of Q.E.D.

At 1 loop we consider the diagram:



$$\equiv i\Pi^{\mu\nu}_{1\,\text{Loop}}(q)$$

The amplitude is:

$$i\Pi^{\mu\nu}_{1\,\text{LOOP}}(q) = (-1)(ie)^2 \int \frac{d^4K}{(2\pi)^4} \frac{tr\left[\gamma^\mu i(\not{K}+m)\gamma^\nu i(\not{K}+\not{q}+m)\right]}{(K^2-m^2+i\varepsilon)\left[(K+q)^2-m^2+i\varepsilon\right]}$$

we now compute this loop, following a number of steps:

1) Simplify the integrand

$$i\Pi^{\mu\nu}_{1\,\text{LOOP}}(q) = -e^2 \int \frac{d^4K}{(2\pi)^4} \frac{tr\left[\gamma^\mu i(\not{K}+m)\gamma^\nu i(\not{K}+\not{q}+m)\right]}{(K^2-m^2+i\varepsilon)\left[(K+q)^2-m^2+i\varepsilon\right]} =$$

$$= -4e^2 \int \frac{d^4K}{(2\pi)^4} \frac{tr\left[\gamma^\mu \not{K}\gamma^\nu(\not{K}+\not{q})\right] + m^2 tr(\gamma^\mu\gamma^\nu)}{(K^2-m^2+i\varepsilon)\left[(K+q)^2-m^2+i\varepsilon\right]} =$$

$$= -4e^2 \int \frac{d^4K}{(2\pi)^4} \frac{\left[K_\rho(K+q)_\sigma(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}) + m^2 g^{\mu\nu}\right]}{(K^2-m^2+i\varepsilon)\left[(K+q)^2-m^2+i\varepsilon\right]} =$$

$$= -4e^2 \int \frac{d^4K}{(2\pi)^4} \frac{K^\mu(K+q)^\nu - K\cdot(K+q)g^{\mu\nu} + K^\nu(K+q)^\mu + m^2 g^{\mu\nu}}{(K^2-m^2+i\varepsilon)\left[(K+q)^2-m^2+i\varepsilon\right]} =$$

$$= -4e^2 \int \frac{d^4K}{(2\pi)^4} \frac{K^\mu(K+q)^\nu + K^\nu(K+q)^\mu - g^{\mu\nu}\left[K\cdot(K+q) - m^2\right]}{(K^2-m^2+i\varepsilon)\left[(K+q)^2-m^2+i\varepsilon\right]}$$

2) <u>We introduce the so called Feynman parameter</u>

We use the formula:

$$\boxed{\frac{1}{AB} = \int_0^1 dx \frac{1}{\left[(1-x)A + xB\right]^2}}$$

"x" is known as a Feynman parameter

we apply this formula to the propagator denominators:

$$\frac{1}{(K^2-m^2+i\varepsilon)\left[(K+q)^2-m^2+i\varepsilon\right]} = \int_0^1 dx \frac{1}{\left\{(1-x)(K^2-m^2+i\varepsilon) + x\left[(K+q)^2-m^2+i\varepsilon\right]\right\}^2}$$

Let's focus on the denominator:

$$\begin{aligned} \text{DEN} &= (1-x)(k^2-m^2+i\varepsilon) + x(k^2+q^2+2k\cdot q-m^2+i\varepsilon) = \\ &= k^2(1-x+x) + xq^2 + 2xk\cdot q - m^2(1-x+x) + i\varepsilon(1-x+x) = \\ &= k^2 + xq^2 + 2xk\cdot q - m^2 + i\varepsilon + x^2q^2 - x^2q^2 = \\ &= (k+xq)^2 + xq^2(1-x) - m^2 + i\varepsilon \end{aligned}$$

$$\longrightarrow \quad \frac{1}{(k^2-m^2+i\varepsilon)\left[(k+q)^2-m^2+i\varepsilon\right]} = \int_0^1 dx \, \frac{1}{\left\{(k+xq)^2 + xq^2(1-x) - m^2 + i\varepsilon\right\}^2}$$

$$\longrightarrow \quad \boxed{i\Pi^{\mu\nu}_{1\,LOOP}(q) = -4e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \, \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}\left[k\cdot(k+q) - m^2\right]}{\left\{(k+xq)^2 + xq^2(1-x) - m^2 + i\varepsilon\right\}^2}}$$

3) Dimensional regularization

Before proceding we need to regularize the integral to make it finite .

- One idea could be to make a cutoff ⟶ this would be not compatible with Gauge invariance
- In order to do that we adopt the so called dimensional regularization:

we lower the number of space-time dimension from $4 \longrightarrow d = 4-\varepsilon < 4$

If we lower the dimension of space-time we also lower "D" and the integral will converge. This is a math trick we use to manipulate the integrals, at the end we'll go back to $\varepsilon \to 0$.
The integral becomes:

$$-4e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \, \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}\left[k\cdot(k+q) - m^2\right]}{\left[(k+xq)^2 + xq^2(1-x) - m^2 + i\varepsilon\right]^2}$$

↳ we take d small enough such that the integral converges

4) Shift in the integration variable

We set $\boxed{k^\mu + xq^\mu \equiv \ell^\mu}$ and define $\boxed{\Delta \equiv m^2 - x(1-x)q^2}$. The numerator becomes:

$$\begin{aligned} & k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}\left[k\cdot(k+q) - m^2\right] = \\ = & \left[(\ell^\mu - xq^\mu)(\ell^\nu + (1-x)q^\nu) + (\mu \longleftrightarrow \nu)\right] - g^{\mu\nu}\left[(\ell - xq)\cdot(\ell + (1-x)q) - m^2\right] = \\ = & \left[\ell^\mu\ell^\nu + (1-x)\ell^\mu q^\nu - xq^\mu\ell^\nu - x(1-x)q^\mu q^\nu + (\mu \longleftrightarrow \nu)\right] - g^{\mu\nu}\left[\ell^2 + \ell\cdot q(1-x) - x\ell\cdot q - x(1-x)q^2 - m^2\right] \equiv \mathcal{N}^{\mu\nu} \end{aligned}$$

$$\longrightarrow \quad \boxed{-4e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \, \frac{\mathcal{N}^{\mu\nu}}{(\ell^2 - \Delta + i\varepsilon)^2}}$$

All terms in $\mathcal{N}^{\mu\nu}$ that are linear in "ℓ" vanish because of parity ($\ell^\mu \to -\ell^\mu$). With this argument $\mathcal{N}^{\mu\nu}$ drastically simplifies in:

$$\boxed{\mathcal{N}^{\mu\nu} = 2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}\left[m^2 + x(1-x)q^2\right]}$$

The loop integral becomes:

$$-4e^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\,\frac{2\ell^\mu\ell^\nu-g^{\mu\nu}\ell^2-2x(1-x)q^\mu q^\nu+g^{\mu\nu}\left[m^2+x(1-x)q^2\right]}{(\ell^2-\Delta+i\varepsilon)^2}$$

5) Reduction of tensor structures

Consider the integral: $\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta+i\varepsilon)}$ this is a tensor integral which is moreover only function of $\Delta=m^2-x(1-x)q^2$ that is a scalar quantity.

We thus expect the following result:

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta+i\varepsilon)^2}=\mathcal{J}(\Delta)\,g^{\mu\nu}$$

with $g^{\mu\nu}$ that accounts for the tensor structure

We solve for $\mathcal{J}(\Delta)$: we write (multiplying both sides with $g_{\mu\nu}$):

$$\int\frac{d^d\ell}{(2\pi)^d}\frac{g_{\mu\nu}\ell^\mu\ell^\nu}{(\ell^2-\Delta+i\varepsilon)^2}=\mathcal{J}(\Delta)\,g_{\mu\nu}\,g^{\mu\nu}=d\cdot\mathcal{J}(\Delta)$$

from which we can extract: $\mathcal{J}(\Delta)=\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2/d}{(\ell^2-\Delta+i\varepsilon)^2}$ and arrive at:

$$\boxed{\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta+i\varepsilon)^2}=\int\frac{d^d\ell}{(2\pi)^d}\frac{g^{\mu\nu}\,\ell^2/d}{(\ell^2-\Delta+i\varepsilon)^2}}$$

Our original integral now reads:

$$-4e^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\,\frac{2\ell^2g^{\mu\nu}/d-g^{\mu\nu}\ell^2-2x(1-x)q^\mu q^\nu+g^{\mu\nu}\left[m^2+x(1-x)q^2\right]}{(\ell^2-\Delta+i\varepsilon)^2}$$

$$=-4e^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\,\frac{-(1-2/d)\ell^2g^{\mu\nu}-2x(1-x)q^\mu q^\nu+\left[m^2+x(1-x)q^2\right]g^{\mu\nu}}{(\ell^2-\Delta+i\varepsilon)^2}$$

6) Perform a Wick rotation

Let us introduce the idea of Wick rotation in general, consider the integral:

$$\boxed{\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^{2n}}{(\ell^2-\Delta+i\varepsilon)^m}\quad\text{with}\quad\Delta>0}$$

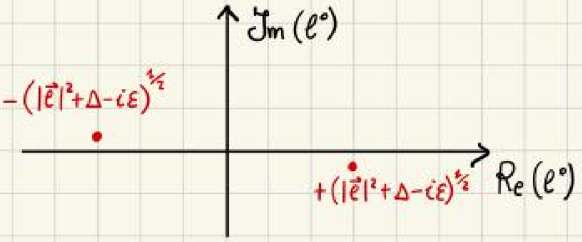
we write it as follows:

$$\int\frac{d^{d-1}\vec{\ell}}{(2\pi)^d}\int_{-\infty}^{+\infty}d\ell^0\,\frac{\ell^{2n}}{\left[(\ell^0)^2-|\vec{\ell}|^2-\Delta+i\varepsilon\right]^m}$$

⟶ we focus on the ℓ^0-integral.

With respect to ℓ^0, the integrand has poles at $(\ell^0)^2=|\vec{\ell}|^2+\Delta-i\varepsilon$ ⟶ $\boxed{\ell^0=\pm\left(|\vec{\ell}|^2+\Delta-i\varepsilon\right)^{1/2}}$

The poles are located as in the following figure:



To be convinced we rewrite:

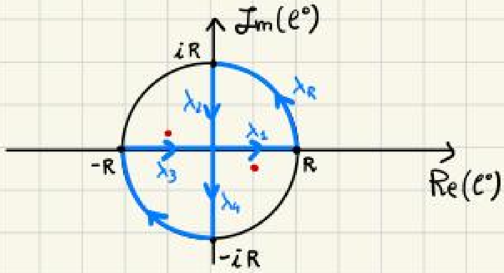
$$\ell^0=\pm\sqrt{|\vec{\ell}|^2+\Delta}\left(1-\frac{i\varepsilon}{|\vec{\ell}|^2+\Delta}\right)^{\frac{1}{2}}\simeq+\left(|\vec{\ell}|^2+\Delta\right)^{\frac{1}{2}}\left(1-\frac{i\varepsilon}{2(|\vec{\ell}|^2+\Delta)}\right)=\pm\left(\sqrt{|\vec{\ell}|^2+\Delta}-\frac{i\varepsilon}{2\sqrt{|\vec{\ell}|^2+\Delta}}\right)=\pm\left(|\vec{\ell}|^2+\Delta\right)^{\frac{1}{2}}\mp\frac{i\varepsilon}{2(|\vec{\ell}|^2+\Delta)^{\frac{1}{2}}}$$

we are interested in the integral along the real line, we use the Cauchy theorem:

We take the contour as in the figure and consider the limit $R\longrightarrow\infty$.

Schematically we write:

$$\boxed{\begin{array}{l}\lim\limits_{R\to\infty}\int_{\lambda_1}d\ell^0+\lim\limits_{R\to\infty}\int_{\lambda_2}d\ell^0=0\\[2ex]\lim\limits_{R\to\infty}\int_{\lambda_3}d\ell^0+\lim\limits_{R\to\infty}\int_{\lambda_4}d\ell^0=0\end{array}}$$



With parametrization $\begin{cases}\lambda_2(\ell^0)=i\ell^0, & R\geqslant\ell^0\geqslant0\\ \lambda_4(\ell^0)=i\ell^0, & 0\geqslant\ell^0\geqslant-R\end{cases}$ we find:

$$\boxed{\begin{array}{l}\int_0^\infty d\ell^0 f(\ell^0)+i\int_\infty^0 d\ell^0 f(i\ell^0)=0\\[2ex]\int_0^\infty d\ell^0 f(\ell^0)+i\int_0^{-\infty} d\ell^0 f(i\ell^0)=0\end{array}}\rightarrow\boxed{\int_{-\infty}^{+\infty}d\ell^0 f(\ell^0)=i\int_{-\infty}^{\infty}d\ell^0 f(i\ell^0)}$$

we use this formula in our integral

Therefore:

$$\int\frac{d^{d-1}\vec{\ell}}{(2\pi)^d}\int_{-\infty}^{+\infty}d\ell^0\frac{\left[(\ell^0)^2-|\vec{\ell}|^2\right]^n}{\left[(\ell^0)^2-|\vec{\ell}|^2-\Delta+i\varepsilon\right]^m}\overset{\text{"Wick rotation"}}{\overset{\downarrow}{=}}\int\frac{d^{d-1}\vec{\ell}}{(2\pi)^d}\,i\int_{-\infty}^{+\infty}d\ell^0\frac{(-(\ell^0)^2-|\vec{\ell}|^2)^n}{(-1)^m\left[(\ell^0)^2-|\vec{\ell}|^2-\Delta+\cancel{i\varepsilon}\right]^m}=$$

↖ we drop $i\varepsilon$ since we are now far from the poles

$$=\int\frac{d^{d-1}\vec{\ell}}{(2\pi)^d}\,i\int_{-\infty}^{+\infty}d\ell^0\frac{(-1)^n((\ell^0)^2+|\vec{\ell}|^2)^n}{(-1)^m\left[(\ell^0)^2+|\vec{\ell}|^2+\Delta\right]^m}$$

The signature is now Euclidean. We introduce the following d-dimensional vector:

$$\boxed{\vec{\ell}_E\equiv(\ell^0;\vec{\ell}),\ |\vec{\ell}_E|^2=(\ell^0)^2+|\vec{\ell}|^2\equiv\ell_E^2}$$

and we arrive at the integral:

$$\boxed{\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^{2n}}{(\ell^2-\Delta+i\varepsilon)^m}=(-1)^{n-m}\,i\int\frac{d^d\vec{\ell}_E}{(2\pi)^d}\frac{|\vec{\ell}_E|^{2n}}{(|\vec{\ell}_E|^2+\Delta)^m}}$$

we apply this formula to our previous integral, arriving at:

$$\boxed{i\,\Pi^{\mu\nu}_{1\,\text{LOOP}}(q) = -4e^2 i\int_0^1 dx \int \frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{(1-2/d)\ell_E^2 g^{\mu\nu} - 2x(1-x)q^\mu q^\nu + \left[m^2+x(1-x)q^2\right]g^{\mu\nu}}{(\ell_E^2+\Delta)^2}}$$

7) d-DIM Euclidean integral

Consider the integral: $\displaystyle\int \frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{(\ell_E^2)^n}{(\ell_E^2+\Delta)^m}$

We can use spherical symmetry in d-dimension (euclidean):

$$\int \frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{(\ell_E^2)^n}{(\ell_E^2+\Delta)^m} = \int\frac{d\Omega_d}{(2\pi)^d}\int_0^\infty d\ell_E\;\ell_E^{d-1}\,\frac{\ell_E^{2n}}{(\ell_E^2+\Delta)^m} =$$

n.b. $\boxed{\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}}$ Surface area of a d-dim sphere

$$= \frac{1}{(2\pi)^d}\,\frac{(2\pi)^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty d\ell_E\;\frac{\ell_E^{d-1+2n}}{(\ell_E^2+\Delta)^m}$$

Let's focus on the radial part:

$$\int_0^\infty d\ell_E\;\frac{\ell_E\,\ell_E^{d-2+2n}}{(\ell_E^2+\Delta)^m} = \frac{1}{2}\int_0^\infty d(\ell_E^2)\,\frac{(\ell_E^2)^{n+d/2-1}}{(\ell_E^2+\Delta)^m} \overset{(*)}{=}$$

we introduce: $x \equiv \dfrac{\Delta}{\ell_E^2+\Delta} \longrightarrow \ell_E^2 = \dfrac{\Delta(1-x)}{x}$; $d(\ell_E^2) = -\dfrac{\Delta}{x^2}dx$, therefore:

$$\overset{(*)}{=} \frac{1}{2}\int_1^0 dx\left(\frac{-\Delta}{x^2}\right)\left[\frac{\Delta^{n+\frac{d}{2}-1}(1-x)^{n+\frac{d}{2}-1}}{x^{n+\frac{d}{2}-1}}\right]\frac{x^m}{\Delta^m} =$$

$$= \frac{1}{2}\int_0^1 dx\;\Delta^{n+\frac{d}{2}-1+1-m}(1-x)^{n+\frac{d}{2}-1}x^{m-2-n-\frac{d}{2}+1} =$$

$$= \frac{1}{2}\left(\frac{1}{\Delta}\right)^{m-n-\frac{d}{2}}\int_0^1 dx\,(1-x)^{n+\frac{d}{2}-1}x^{m-n-1-\frac{d}{2}} \overset{(*)}{=}$$

this integral is of the type:

$$\int_0^1 dx\,x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{with} \quad \begin{cases} \alpha-1 = m-n-1-\frac{d}{2} \longrightarrow \alpha = m-n-d/2 \\ \beta-1 = n+\frac{d}{2}-1 \longrightarrow \beta = n+d/2 \\ \alpha+\beta = m-n-\frac{d}{2}+n+\frac{d}{2} = m \end{cases}$$

$$\overset{(*)}{=} \frac{1}{2}\left(\frac{1}{\Delta}\right)^{m-n-\frac{d}{2}}\frac{\Gamma\left(m-n-\frac{d}{2}\right)\Gamma\left(n+\frac{d}{2}\right)}{\Gamma(m)}$$

Finally going back to the full integral we get:

$$\int\frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{(\ell_E^2)^n}{(\ell_E^2+\Delta)^m} = \frac{1}{(2\pi)^d}\,\frac{(2\pi)^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\,\frac{1}{2}\left(\frac{1}{\Delta}\right)^{m-n-\frac{d}{2}}\frac{\Gamma\left(m-n-\frac{d}{2}\right)\cdot\Gamma\left(n+\frac{d}{2}\right)}{\Gamma(m)} =$$

$$= \frac{1}{(4\pi)^{d/2}}\,\frac{1}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{1}{\Delta}\right)^{m-n-\frac{d}{2}}\cdot\frac{\Gamma\left(m-n-\frac{d}{2}\right)\Gamma\left(n+\frac{d}{2}\right)}{\Gamma(m)}$$

$$\longrightarrow \boxed{\int \frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{(\ell_E^2)^n}{(\ell_E^2+\Delta)^m} = \frac{1}{(4\pi)^{d/2}}\,\frac{1}{\Gamma(\frac{d}{2})}\left(\frac{1}{\Delta}\right)^{m-n-\frac{d}{2}}\cdot\frac{\Gamma(m-n-\frac{d}{2})\,\Gamma(n+\frac{d}{2})}{\Gamma(m)}}$$

Notable cases:

- $\boxed{n=0,\ m=2}$

$$\longrightarrow \int \frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{1}{(\ell_E^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\cdot\frac{\Gamma(2-\frac{d}{2})\,\cancel{\Gamma(\frac{d}{2})}}{\cancel{\Gamma(\frac{d}{2})}\,\underset{=1}{\cancel{\Gamma(2)}}}$$

n.b. $z\,\Gamma(z) = \Gamma(z+1)$

$$\longrightarrow \boxed{\int \frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{1}{(\ell_E^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\cdot\Gamma(2-\tfrac{d}{2})}$$

Finally we can apply these results to our case

Esercizio : After_dinner_exercise (beautiful exercise on quantum gravity)

$$-4e^2 i\int_0^1 dx\left\{g^{\mu\nu}\left(1-\frac{2}{d}\right)\int\frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^2}+\left[2x(1-x)\,q^\mu q^\nu+(m^2+x(1-x)q^2)g^{\mu\nu}\right]\cdot\int\frac{d^d\vec{\ell}_E}{(2\pi)^d}\,\frac{1}{(\ell_E^2+\Delta)^2}\right\}$$

let's plug-in the expression found:

$$=-4e^2 i\int_0^1 dx\,\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}\left\{\left(1-\frac{2}{d}\right)g^{\mu\nu}\,\frac{d}{2}\,\frac{\Delta}{(1-\frac{d}{2})}+\left[-2x(1-x)\,q^\mu q^\nu+m^2g^{\mu\nu}+x(1-x)q^2g^{\mu\nu}\right]\right.$$

I focus on curly brackets:

$$\left(1-\frac{2}{d}\right)g^{\mu\nu}\,\frac{\cancel{d}}{\cancel{2}}\,\frac{\Delta}{\frac{\cancel{d}}{\cancel{2}}\left(\frac{2}{d}-1\right)}-2x(1-x)\,q^\mu q^\nu+m^2g^{\mu\nu}+x(1-x)\,q^2g^{\mu\nu} =$$

$$=-g^{\mu\nu}\Delta-2x(1-x)\,q^\mu q^\nu+m^2g^{\mu\nu}+x(1-x)\,q^2g^{\mu\nu}=$$

$$=\cancel{-g^{\mu\nu}m^2}+g^{\mu\nu}x(1-x)\,q^2-2x(1-x)\,q^\mu q^\nu+\cancel{m^2g^{\mu\nu}}+x(1-x)\,q^2g^{\mu\nu}=$$

$$=2x(1-x)\left(g^{\mu\nu}q^2-q^\mu q^\nu\right)$$

$$\longrightarrow \overset{(*)}{=} -4e^2 i\int_0^1 dx\,\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}\cdot 2x(1-x)\left(q^2g^{\mu\nu}-q^\mu q^\nu\right)$$

$$\longrightarrow \boxed{i\,\Pi^{\mu\nu}_{1L}(q)=\left(q^2g^{\mu\nu}-q^\mu q^\nu\right)\Pi_{1L}(q^2) \qquad \Pi^{\mu\nu}_{1L}(q^2)=-\frac{8e^2}{(4\pi)^{d/2}}\int_0^1 dx\ x(1-x)\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}}$$

This expression must diverge in the limit of $d=4$. and indeed it's true.

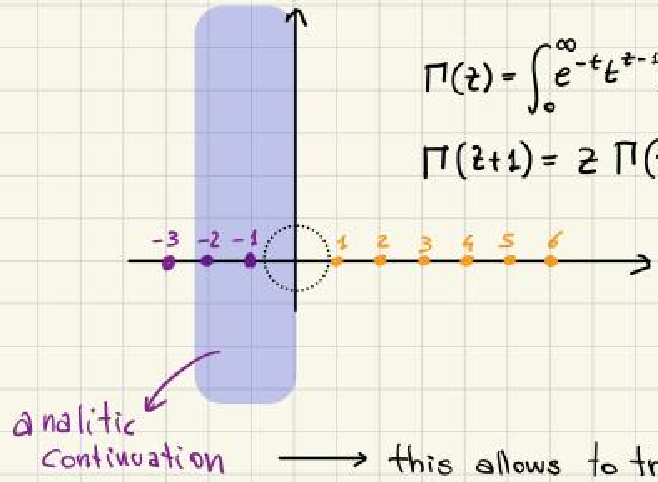
$$z=0,-1,-2, \qquad z=2-\frac{d}{2} \longrightarrow \text{if } d=4 \quad z=0$$

$\Gamma(z)$ is perfect analitical for $z>0$ which means $2-\frac{d}{2}>0 \rightarrow d<4$ This agree on the fact that if we lower dimension we get a not divergent object.

Properties of $\Gamma(z)$

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$$

$$\Gamma(z+1)=z\,\Gamma(z)$$



$\longrightarrow$ this allows to treat z (d) as a continuous variable

We can make a Loran exp:

$$d=4-2\varepsilon \longrightarrow \varepsilon=2-\frac{d}{2}$$

1) $\Gamma(2-\frac{d}{2})=\Gamma(\varepsilon) \overset{\text{Loran exp.}}{=} \frac{1}{\varepsilon}-\gamma+o(\varepsilon)$ $\qquad \gamma$: Euler Mascheroni constant

2) $\frac{1}{(4\pi)^{\frac{d}{2}}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}=\frac{1}{(4\pi)^2}\left(\frac{4\pi}{\Delta}\right)^{\varepsilon} \simeq \frac{1}{(4\pi)^2}\cdot\left[1+\varepsilon\log\left(\frac{4\pi}{\Delta}\right)+o(\varepsilon^2)\right]$

$\hookrightarrow a^{\varepsilon}\simeq 1+\varepsilon\log a+o(\varepsilon^2)$

$$\longrightarrow \Pi_{1L}(q^2)=-\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left[1+\varepsilon\log\left(\frac{4\pi}{\Delta}\right)+o(\varepsilon^2)\right]\left(\frac{1}{\varepsilon}-\gamma+o(\varepsilon)\right)$$

$$\longrightarrow \Pi_{1L}(q^2)=-\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left[\frac{1}{\varepsilon}-\gamma+\log\left(\frac{4\pi}{\Delta}\right)\right]$$

Now we can start to isolate the origin of the divergence thanks to DIMREG. Moreover remember that: $[\Delta]=[M]^2$, the argument of log has to be dimensionless. The point is that as soon as we change the dim the QED coupling is no more dimensionless:

$$[e]=[M]^{2-\frac{d}{2}}=[M]^{\varepsilon} \qquad \text{(for } d=4\ (\varepsilon=0) \text{ it is dimensionless)}$$

Check: $[\mathcal{L}]=[M]^d$; $\begin{cases}[\psi]^2[M]\overset{!}{=}[M]^d \longrightarrow [\psi]=[M]^{\frac{d-1}{2}}\\ [A^\mu]^2[M]^2\overset{!}{=}[M]^d \longrightarrow [A^\mu]=[M]^{\frac{d}{2}-1}\end{cases}$

$\longrightarrow$ (considering $e\bar{\psi}A_\mu\psi$ $\rightarrow$ $[e]\cdot[M]^{d-1}[M]^{\frac{d}{2}-1}\overset{!}{=}[M]^d \longrightarrow [e]=[M]^{2-\frac{d}{2}}$

We can redefine: $e\longrightarrow\mu^{\varepsilon}e$ (now e is dimensionless and μ (renormalization scale) is dimensionfull) μ keeps track of the dimensions. It is an unphysical quantity i.e. at the end its contribution must cancel out.

$$\frac{1}{(4\pi)^{\frac{d}{2}}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2}\left(\frac{4\pi\mu^2}{\Delta}\right)^{\varepsilon} \simeq \frac{1}{(4\pi)^2}\cdot\left[1+\varepsilon\log\left(\frac{4\pi\mu^2}{\Delta}\right)+o(\varepsilon^2)\right]$$

$$\longrightarrow \boxed{\begin{aligned} i\,\Pi^{\mu\nu}_{1L}(q) &= (q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi_{1L}(q^2) \\ \Pi_{1L}(q^2) &= -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left[\frac{1}{\varepsilon}-\gamma+\log\left(\frac{4\pi\mu^2}{\Delta}\right)\right]\end{aligned}}$$

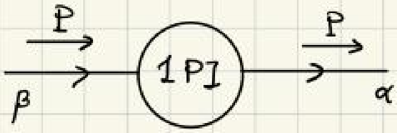
Regularised photon self-energy

This divergence is contained in $\Pi_{1L}(q^2=0)$ (diverges as $1/\varepsilon$) all the rest will not diverge, indeed:

$$\Pi_{1\,Loop}(q^2) - \Pi_{1\,Loop}(q^2=0) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2}{m^2-x(1-x)q^2}\right)$$

ELECTRON SELF-ENERGY

Now we can try to do the same for the "other object": electron self energy let's consider:



$\equiv i\left[\Sigma(p)\right]_{\alpha\beta}$ D=1

The amplitude would be

$$i\mathcal{M} = \bar{u}_\alpha(p)\, i\,\Sigma(p)_{\alpha\beta}\, u_\beta(p) \qquad \text{(for a more complicated diagram)}$$

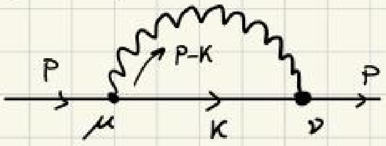
but we consider the amputated amplitude: $i\mathcal{M} = i\,\Sigma(p)_{\alpha\beta}$

I can write in general this amplitude as:

$$\boxed{i\left[\Sigma(p)\right]_{\alpha\beta} = A(p^2)\,\delta_{\alpha\beta} + B(p^2)\,p_\mu\,(\gamma^\mu)_{\alpha\beta}} + \,?\qquad \ast$$

- $C(p^2)\,p_\mu p_\nu\,(\gamma^\mu\gamma^\nu)_{\alpha\beta} \sim C(p^2)\,p^2\,\delta_{\alpha\beta}$
- $D(p^2)\,p_\mu p_\nu p_\rho\,(\gamma^\mu\gamma^\nu\gamma^\rho)_{\alpha\beta} \sim D(p^2)\,p_\rho\,(\gamma^\rho)_{\alpha\beta}$
- $E(p^2)\,p_\mu\,(\gamma^\mu)_{\alpha\beta}\,\gamma^5$ violates parity (QED does not do it)

Let's consider the following diagram:



$$i\,\Sigma_{1\,Loop}(p) = (+ie)^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu i(\not{k}+m)\gamma^\nu}{(k^2-m^2+i\varepsilon)}\,\frac{(-i g_{\mu\nu})}{[(p-k)^2+i\varepsilon]} =$$

$$= (+ie^2)\int\frac{d^4k}{(2\pi)^4}\,\frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(k^2-m^2+i\varepsilon)[(p-k)^2+i\varepsilon]} =$$

$$= (+ie)^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \quad \frac{\gamma^\mu (\not{k} + m) \gamma_\mu}{\left[(k - xp)^2 - x(1-x)p^2 - (1-x)m^2 + i\varepsilon\right]^2}$$

Now I use DIM. REG. ; I can define: $k - xp \equiv \ell$, $k = \ell + xp$, $\Delta \equiv m^2(1-x) - x(1-x)p^2$

$$\longrightarrow \quad (+ie)^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{\gamma^\mu (\not{\ell} + x\not{p} + m)\gamma_\mu}{(\ell^2 - \Delta + i\varepsilon)^2}$$

Now I use that: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$; $g^{\mu\nu} g_{\mu\nu} = d$; $\gamma^\mu \gamma_\mu = d \longrightarrow \gamma^\mu \gamma^\nu \gamma_\mu = (2-d)\gamma^\nu$

$$= (+ie)^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(2-d)\,x\not{p} + md}{(\ell^2 - \Delta + i\varepsilon)}$$

Now we perform a Wick rotation:

$$\ell^0 \longrightarrow i\ell_E^0 \qquad \vec{\ell}_E = (\ell_E^0, \vec{\ell})$$

$$= (ie)^2 \mu^{4-d} i \int_0^1 dx \left[(2-d)\,x\not{p} + dm\right] \int \frac{d^d\vec{\ell}_E}{(2\pi)^d} \cdot \frac{1}{(\ell_E^2 + \Delta)^2}$$

$$= (ie)^2 \mu^{4-d} i \int_0^1 dx \left[(2-d)\,x\not{p} + dm\right] \cdot \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Delta^{2-d/2}}$$

Remember that $d = 4 - 2\varepsilon \longrightarrow \varepsilon = 2 - \frac{d}{2}$. Therefore: $(2-d) = -2(1-\varepsilon)$; $\frac{d}{2} = 2 - \varepsilon$

$$= (ie)^2 i \int_0^1 dx \left[-2(1-\varepsilon)x\not{p} + (4 - 2\varepsilon)m\right] \frac{(4\pi)^\varepsilon}{(4\pi)^2} \frac{\Gamma(\varepsilon)}{\Delta^\varepsilon} \left(\frac{4\pi\mu^2}{\Delta}\right)^\varepsilon$$

$$= \frac{(ie)^2}{(4\pi)^2} \cdot \int_0^1 dx \left[-2(1-\varepsilon)x\not{p} + (4-2\varepsilon)m\right] \cdot \left[\frac{1}{\varepsilon} - \gamma + \log\left(\frac{4\pi^{3}\mu^2}{\Delta}\right)\right]$$

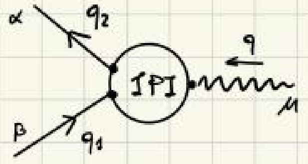
This expression is totally compatible with * if :

$$\boxed{\begin{aligned} A(p^2) &= \frac{(ie)^2}{(4\pi)^2} \int_0^1 dx \; 4m \left[\frac{1}{\varepsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{\Delta}\right) - \frac{1}{2}\right] \\ B(p^2) &= \frac{(ie)^2}{(4\pi)^2} \int_0^1 dx \; (-2x) \left[\frac{1}{\varepsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{\Delta}\right) - \frac{1}{2}\right] \end{aligned}}$$

If we take the limit $\varepsilon \to 0$ these 2 fonctions diverge (U.V.) indipendently.

VERTEX CORRECTION

Let's consider the vertex function :



$$\equiv ie\left(\Gamma^\mu(q_1, q_2)\right)_{\alpha\beta}$$

$q_1 + q = q_2$

$\rightarrow q = q_2 - q_1$

What about $\Gamma^{\mu}(q_1,q_2)_{\alpha\beta} = ?$ It's a mess in general. In general it is:

$$\Gamma^{\mu}(q_1,q_2)_{\alpha\beta} = \left(A_1 + A_2 \not{q}_1 + A_3 \not{q}_2 + A_4 \not{q}_1 \not{q}_2\right) \cdot q_1^{\mu}$$

$$+ \left(B_1 + B_2 \not{q}_1 + B_3 \not{q}_2 + B_4 \not{q}_1 \not{q}_2\right) q_2^{\mu}$$

$$+ \left(C_1 \gamma^{\mu} + C_2 \gamma^{\mu} \not{q}_1 + C_3 \gamma^{\mu} \not{q}_2 + C_4 \not{q}_2 \gamma^{\mu} \not{q}_1\right)$$

Where A_i, B_i, C_i are scalar functions of the scalars $(q_1^2, q_2^2, q_1 \cdot q_2)$. n.b. the spinorial form is given by $\not{q}$! We can think for simplicity that the spinors are on the mass shell. Therefore:

$$\bar{u}(q_2)\, ie\, \Gamma^{\mu}(q_1,q_2)\, u(q_1)$$

We can use Dirac equation : $\begin{cases} \not{q}_1 u(q_1) = m\, u(q_1) \\ \bar{u}(q_2) \not{q}_2 = m\, \bar{u}(q_2) \end{cases}$

$$\longrightarrow \Gamma^{\mu}(q_1,q_2)_{\alpha\beta} = \left(A_1 + A_2 m + A_3 m + A_4 m^2\right) \cdot q_1^{\mu}$$

$$+ \left(B_1 + B_2 m + B_3 m + B_4 m^2\right) q_2^{\mu}$$

$$+ \left(C_1 \gamma^{\mu} + C_2 \gamma^{\mu} m + C_3 m \gamma^{\mu} + C_4 m^2 \gamma^{\mu}\right)$$

This means that all the functions are function of (m^2, q^2)

$$= A(q^2)\, q_1^{\mu} + B(q^2)\, q_2^{\mu} + C(q^2)\, \gamma^{\mu}$$

This is not the end of the story : there's a photon. This means that:

Ward identity: $q_{\mu} \Gamma^{\mu}(q_1,q_2) = 0 \quad ; \quad q = q_2 - q_1$

Therefore:

$$\overset{m-m=0 \text{ (Applying Dirac)}}{}$$

$$= A(q^2) \cdot (q_2 - q_1) \cdot q_1 + B(q^2)(q_2 - q_1) \cdot q_2 + \cancel{C(q^2)(\not{q}_2 - \not{q}_1)}$$

$$= A(q^2)(q_2 \cdot q_1 - m^2) + B(q^2)(m^2 - q_1 \cdot q_2) \overset{!}{=} 0 \implies A(q^2) = B(q^2)$$

$$\longrightarrow \Gamma^{\mu}(q_1,q_2)_{\alpha\beta} = A(q^2)(q_1 + q_2)^{\mu} + C(q^2)\gamma^{\mu}$$

$A(q^2)$ and $C(q^2)$ are called form factors. Now using the Gordon identity:

$$2m\, \bar{u}(q_2) \gamma^{\mu} u(q_1) = (q_2 + q_1)^{\mu}\, \bar{u}(q_2) u(q_1) + i (q_2 - q_1)_{\nu}\, \bar{u}(q_2)\, \sigma^{\mu\nu} u(q_1) \quad \text{where} \quad \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]$$

CO.RE. Standard Model : serie di formule utili in PARTICLE PHYSICS (paper)
(Compendium of Relations)

The final result is the following:

$$\boxed{\Gamma^\mu(q_1,q_2) = F_1(q^2)\gamma^\mu + \frac{i\,\sigma^{\mu\nu}}{2m}F_2(q^2)\,q_\nu \qquad \begin{array}{ll} F_1 \equiv 2mA + C & \text{(electric form factor)} \\ F_2 \equiv 2m(-1)A & \text{(magnetic form factor)} \end{array}}$$

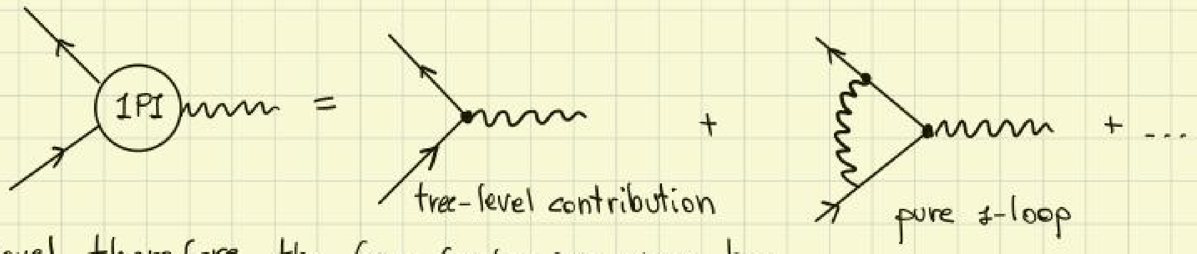
We have the opportunity now to make the following consideration. Remember that this amplitude has $D=0$. If we expand it with respect to the external momentum q we find:

$$\Gamma^\mu(q_1,q_2) \simeq \underbrace{F_1(q^2=0)\gamma^\mu}_{D=0} + \underbrace{\left.\frac{\partial F_1(q^2)}{\partial q^\nu}\right|_{q=0}\gamma^\mu q^\nu + i\,\frac{\sigma^{\mu\nu}}{2m}F_2(q^2=0)\,q^\nu}_{D<0\ \text{(convergent)}} + \dots$$

The only divergent part is contained in $F_1(q^2=0)$

Considerations about the one loop vertex

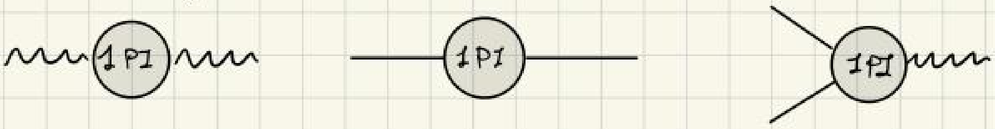
Contrary to the previous case we already have a non-zero contribution to the vertex function at the tree-level



At the 3-level therefore the form factor are given by:

$$\boxed{F_{1,\text{tree}}(q^2) = 1 \quad ; \quad F_{2,\text{tree}}(q^2) = 0 \qquad \forall q^2}$$

So we found that these diagrams are truly divergent objects:



PERTURBATIVE RENORMALIZATION OF QED

Consider the QED Lagrangian:

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\,\overline{\psi}A_\mu\gamma^\mu\psi}$$

we tipically consider "m" and "e" as the electron's mass $m_e \simeq 0.5\,\text{MeV}$ and its charge $e \simeq 0.13$.

Observation 1: in QFT we define the mass of a stable particle to be the pole of its propagator. In the case above we have $\longrightarrow$: $\frac{i}{\not p - m + i\varepsilon}$ hence a pole at $p^2 = m^2$ which is a good def of rest mass in S.R.

Observation 2: in analogy the electric charge is defined computing the potential from the tree-level exchange of a photon and comparing it with the Coulomb potential

These considerations are true only at the tree level and in a free theory.

The "DRESSED" electron propagator

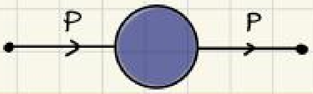
We Know that the free propagator is defined as:

$$\langle 0|T[\psi(x)\overline{\psi}(y)]|0\rangle \equiv \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \cdot \frac{i}{(\not p - m + i\varepsilon)} \leftrightarrow$$

these are free Fields

We are interested now in the computation of the quantity:

$$\boxed{\langle \Omega|T[\psi(x)\overline{\psi}(y)]|\Omega\rangle \leftrightarrow}$$

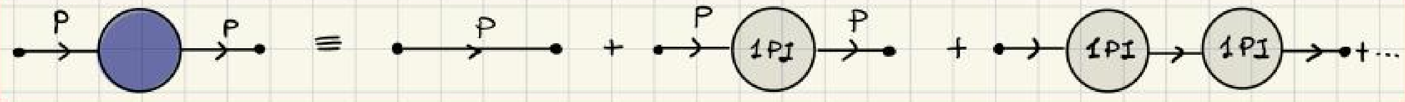


Propagator of the full theory

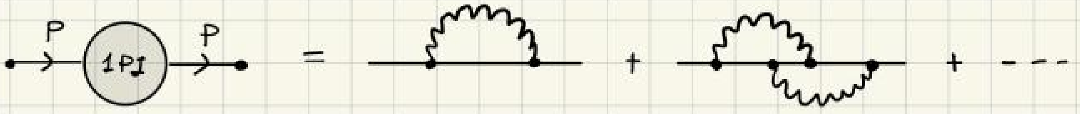
Vacuum of the full theory

fully interacting fields

Where using a diagrammatic approach:

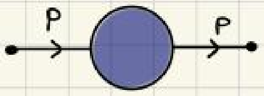


with:



$$\equiv i\,\Sigma(\not p)$$

Mathematically we write :



$$= \frac{i}{\not p - m + i\varepsilon} + \frac{i}{\not p - m + i\varepsilon}\, i\,\Sigma(\not p)\cdot \frac{i}{\not p - m + i\varepsilon} + \frac{i}{\not p - m + i\varepsilon}\, i\,\Sigma(\not p)\cdot \frac{i}{\not p - m + i\varepsilon}\, i\,\Sigma(\not p)\cdot \frac{i}{\not p - m + i\varepsilon} + \cdots =$$

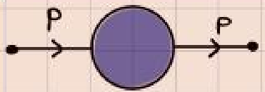
$$= i\left[\frac{1}{\not p - m + i\varepsilon} - \frac{1}{\not p - m + i\varepsilon}\Sigma(\not p)\cdot\frac{1}{\not p - m + i\varepsilon} + \frac{1}{\not p - m + i\varepsilon}\Sigma(\not p)\cdot\frac{1}{\not p - m + i\varepsilon}\Sigma(\not p)\cdot\frac{1}{\not p - m + i\varepsilon} + \cdots\right] \overset{(*)}{=}$$

Now I use the identity:

$$\frac{1}{\hat{x} + \hat{y}} = \frac{1}{\hat{x}} - \frac{1}{\hat{x}}\hat{y}\frac{1}{\hat{x}} + \frac{1}{\hat{x}}\hat{y}\frac{1}{\hat{x}}\hat{y}\frac{1}{\hat{x}} + \cdots$$

$$(*) = i \cdot \frac{1}{\not p - m + i\varepsilon + \Sigma(\not p)}$$

Therefore we got :



$$\boxed{= \frac{i}{\not p - m + i\varepsilon + \Sigma(\not p)}}$$

Dressed electron propagator

2 OBSERVATIONS:

1) The presence of interactions "shift" the mass of the electron respect to "m". (the pole is no more fixed at m). We can write the pole equation :

$$\not p - m + \Sigma(\not p)\Big|_{\not p = m_{phys}} \overset{!}{=} 0$$

$$\longrightarrow \quad \boxed{m_{phys} - m + \Sigma(\not{p} = m_{phys}) = 0} \qquad \underline{\text{Pole equation}}$$

2) *Interactions also change the residue of the pole. Let's expand* Σ *around the pole* m_{phys} :

$$\longrightarrow \frac{i}{\not{p} - m + i\varepsilon + \Sigma(\not{p})} \simeq \frac{i}{\not{p} - m + \left[\Sigma(\not{p} = m_{phys}) + \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p} = m_{phys}} (\not{p} - m_{phys}) + o\,(\not{p} - m_{phys})^2\right] + i\varepsilon} =$$

$$= \frac{i}{\underbrace{\not{p} - m + \Sigma(\not{p} = m_{phys})} + \left[\left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p} = m_{phys}} (\not{p} - m_{phys}) + o\,(\not{p} - m_{phys})^2\right] + i\varepsilon}$$

$$= \frac{i}{\not{p} - m_{phys} + \left[\left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p} = m_{phys}} (\not{p} - m_{phys}) + o\,(\not{p} - m_{phys})^2\right] + i\varepsilon} =$$

$$= \frac{i}{(\not{p} - m_{phys})\left[1 + \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p} = m_{phys}} + o\,(\not{p} - m_{phys})\right] + i\varepsilon} \overset{\text{Laurent expansion}}{\simeq}$$

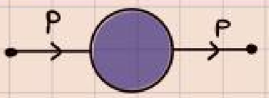
$$\simeq \frac{i Z}{\not{p} - m_{phys} + i\varepsilon} + \text{terms that are regular at } \not{p} = m_{phys}$$

Where Z is the residues (in units of i) and it is given by:

$$\boxed{Z \equiv \frac{1}{1 + \left.\frac{d\Sigma}{d\not{p}}\right|_{m_{phys}}} \neq 1}$$

1 is the residue of the free theory

In conclusion we say that:



$$\simeq \frac{i Z}{\not{p} - m_{phys} + i\varepsilon} + \text{terms that are regular at } \not{p} = m_{phys}$$

$$Z \equiv \frac{1}{1 + \left.\frac{d\Sigma}{d\not{p}}\right|_{m_{phys}}} \quad ; \quad m_{phys} - m + \Sigma(\not{p} = m_{phys}) = 0$$

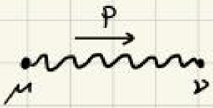
Comment 1: All the above discussion is valid if $\Sigma(\not{p})$ is a well regular function i.e. it mustn't diverge (together with its derivatives). Since we found that $\Sigma(\not{p})$ actually diverges it simply that we're not yet ready to substitute it in the expression above

Comment 2: the parameter "m" in the Lagrangian is not the physical mass of the electron in the full interacting theory.

The "DRESSED" photon propagator

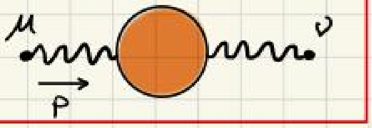
We set a similar computation for the photon. The free propagator is:

$$\langle 0|\, T\left[A^\mu(x)\, A^\nu(y)\right]|0\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} \cdot \frac{i}{p^2+i\varepsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right] \sim$$



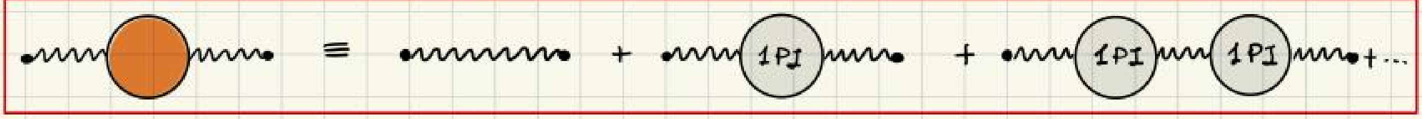
We're interested in the computation of

$$\boxed{\langle \Omega|\, T\left[A^\mu(x)\, A^\nu(x)\right]|\Omega\rangle \longleftrightarrow}$$

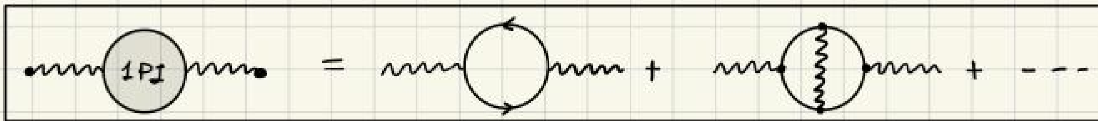


propagator of the full theory.

Where using a diagrammatic approach:



with:



$$\equiv i\,(p^2 g^{\mu\nu} - p^\mu p^\nu)\,\Pi(p^2)$$

We start from some preliminary considerations. We write the free propagator as follows:

$$= \frac{-i}{p^2+i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right) = \frac{-i}{p^2+i\varepsilon}\left[\underbrace{\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)}_{\boxed{P_T^{\mu\nu} \equiv g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}}} + \underbrace{\xi\frac{p^\mu p^\nu}{p^2}}_{\boxed{P_L^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}}}\right]$$

transverse projector

Longitudinal projector

The properties of these 2 projectors are:

$$\boxed{p_\mu P_T^{\mu\nu} = p_\nu P_T^{\mu\nu} = 0}$$

$$\boxed{P_T^{\mu\nu} + P_L^{\mu\nu} = g^{\mu\nu}}$$

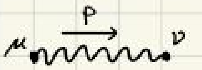
$$\boxed{\begin{array}{l} P_T^2 = P_T \;\; ; \;\; P_L^2 = P_L \\ P_T P_L = P_L P_T = 0 \end{array}}$$

to be clear $P_T^2 = P_T$ means in components:

$$P_T^{\mu\nu} P_{T\,\nu\rho} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(g_{\nu\rho} - \frac{p_\nu p_\rho}{p^2}\right) =$$

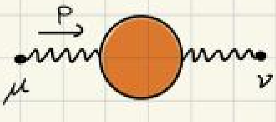
$$= \delta^\mu_\rho - \frac{p^\mu p_\rho}{p^2} - \cancel{\frac{p^\mu p_\rho}{p^2}} + \cancel{\frac{p^2 p^\mu p_\rho}{p^4}} = P_T{}^\mu{}_\rho$$

Therefore:



$$\boxed{= \frac{-i}{p^2+i\varepsilon}\left(P_T^{\mu\nu} + \xi P_L^{\mu\nu}\right)}$$

We thus write:



$$= \frac{(-i)}{p^2+i\varepsilon}\left(P_T^{\mu\nu} + \xi P_L^{\mu\nu}\right) + \frac{(-i)}{p^2+i\varepsilon}\left(P_T^{\mu\nu} + \cancel{\xi P_L^{\mu\nu}}\right)\left[i p^2 (P_T)_{\rho\sigma}\Pi(p^2)\right]\frac{(-i)}{p^2+i\varepsilon}\left(P_T^{\sigma\nu} + \cancel{\xi P_L^{\sigma\nu}}\right) + ---$$

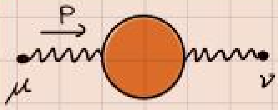
$$= \frac{(-i)}{p^2+i\varepsilon}\left(P_T^{\mu\nu} + \xi P_L^{\mu\nu}\right) + \frac{(-i)}{p^2+i\varepsilon}\, P_T^{\mu\nu}(P_T)_{\rho\sigma}\, P_T^{\sigma\nu}\,\Pi(p^2) + --- =$$

$$= \frac{(-i)}{p^2+i\varepsilon}\left(P_T^{\mu\nu} + \xi P_L^{\mu\nu}\right) + \frac{(-i)}{p^2+i\varepsilon}\, P_T^{\mu\nu}\,\Pi(p^2) + \frac{(-i)}{p^2+i\varepsilon}\, P_T^{\mu\nu}\,\Pi(p^2)^2 + \ldots =$$

$$= \frac{(-i)}{p^2+i\varepsilon} P_T^{\mu\nu}\Big[\underbrace{1+\Pi(p^2)+\Pi(p^2)^2+\dots}_{\text{geometric series, it sums to } \frac{1}{1-\Pi(p^2)}}\Big] - \frac{i}{p^2+i\varepsilon}\,\xi\, P_L^{\mu\nu} =$$

$$= \frac{-i}{p^2+i\varepsilon}\,\frac{P_T^{\mu\nu}}{1-\Pi(p^2)} - \frac{i\xi}{p^2+i\varepsilon} P_L^{\mu\nu}$$

$\longrightarrow$



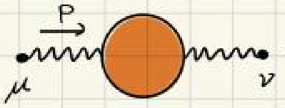
$$= \frac{(-i)\,P_T^{\mu\nu}}{(p^2+i\varepsilon)\left[1-\Pi(p^2)\right]} - \frac{i\xi}{p^2+i\varepsilon} P_L^{\mu\nu}$$

Dressed Photon Propagator

The important points are the following:

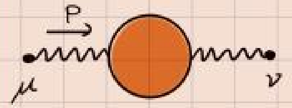
1) Interactions do not shift the photon mass: the pole is still located at $p^2=0$ i.e. the photon remains massless in the full interacting theory.

2) However interactions do change the residue of the pole:



$$= \frac{(-i)\,g^{\mu\nu}}{(p^2+i\varepsilon)\left[1-\Pi(p^2)\right]} + p^\mu p^\nu \text{ terms}$$

It has a pole at $p^2=0$ with residue: $Z \equiv \frac{1}{1-\Pi(p^2=0)}$, so that the Laurent exp. reads:

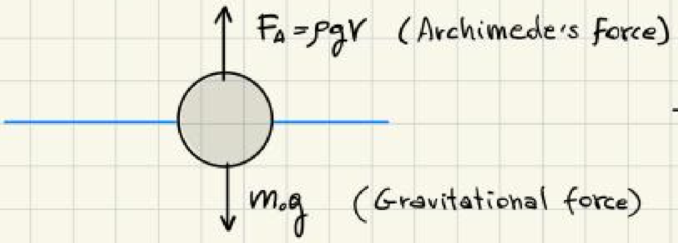


$$= \frac{-i g^{\mu\nu} Z}{(p^2+i\varepsilon)} + \text{terms regular in } q^2 + q^\mu q^\nu \text{ terms}$$

with $Z \equiv \frac{1}{1-\Pi(p^2=0)}$

COMMENT

Suppose we have a rigid sphere of volume V (like a ping-pong ball) immersed in a perfect fluid of density ρ. We assume that $m_0 = \frac{1}{10}\rho V$. If we consider the motion of this sphere in the fluid, an elementary analysis gives:



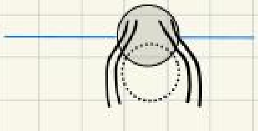
The net force which acts on the ball is:

$$\longrightarrow \quad F = \rho g V - m_0 g = \rho g V - \frac{1}{10}\rho V g = \frac{9}{10}\rho g V = 9 m_0 g$$

$$\longrightarrow \boxed{a = 9g} \quad \text{no sense !}$$

The resolution of the issue is the following. Moving inside the fluid an accelerated body most move some volume of surrounding fluid as it moves through it. Consequently the body has a sort of additional inertia. This phenomenon, indeed can be described by introducing the idea of the "added" mass so that:

$$F = m_0 a \quad \text{becomes} \quad \boxed{F = (m_0 + m_{\text{ADDED}})\,a}$$



In this simple case of a sphere we get:

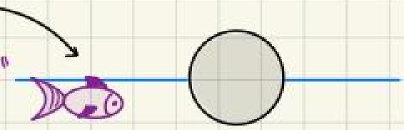
$$m_{\text{ADDED}} = \frac{2}{3}\pi R^3 \rho = \frac{1}{2}\left(\frac{4}{3}\pi R^3\rho\right) = \frac{1}{2} V\rho = \frac{1}{2}(10 m_0) = 5 m_0$$

Consequently:

$$F = (m_o + 5m_o)a = (6\,m_o)a \longrightarrow a = \frac{9m_o g}{6m_o} = \frac{3}{2}g$$

a much more sensible result! If we suppose to be a fish :

it cannot measure
the ball mass m_o (baremass)
it is only sensitive to the "new"
mass (renormalized mass)



THE LSZ REDUCTION FORMULA

IN and OUT FIELDS

Consider a scattering amplitude in terms of asymptotic "in" and "out" states :

$$S_{fi} = \underbrace{{}_{out}\langle\varphi|\psi\rangle_{in}} = \underbrace{\langle\varphi(t=+\infty)|S|\psi(t=-\infty)\rangle}$$

$|\varphi\rangle_{out}$ and $|\psi\rangle_{in}$ are states with well-separated free particles. In the Heisemberg picture they don't depend in time

Description based on the interaction picture

We focus on the Heisemberg picture. We try to construct the "in" and "out" states using the field operators. In the Heisemberg picture, quantum fields do evolve in the according to the full Hamiltonian.
For simplicity we work with scalar fields. $\phi(x)$ is a fully interacting quantum field that evolves according to the full Hamiltonian :

$$\boxed{\phi(\vec{x},t) = e^{itH}\phi(\vec{x},0)e^{-itH}}$$

If we consider the asymptotic past $t \to -\infty$ in which we turn off interactions, we expect that $\phi(x)$ will tend to a free field operator. We would be tempted to write:

$$\boxed{\lim_{t\to-\infty}\phi(x) = \phi_{in}(x) = \int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[a_{in}(\vec{p})e^{-ip\cdot x} + a^\dagger_{in}(\vec{p})e^{ip\cdot x}\right]}$$

free field operator that therefore can be written as a superposition of creation and annihilation operators
Moreover it verifies the usual properties $[a_{in}(\vec{p}), a^\dagger_{in}(\vec{p})] = (2\pi)^3\delta(\vec{p}-\vec{k})$ and it can be used to create particles in the initial state $|\vec{p}\rangle_{in} = \sqrt{2E_p}\,a^\dagger_{in}(\vec{p})|\Omega\rangle$

Comment: here and in the following we indicate with $|\Omega\rangle$ the vacuum of the fully interacting theory while we used $|0\rangle$ for the vacuum of the free theory. Formally we have :

$$\boxed{\begin{array}{l}|0\rangle \text{ ground state of free Hamiltonian } H_0 \\ |\Omega\rangle \text{ ground state of } H = H_0 + H_{int}\end{array}}$$

Comment:
The external states are physical states, consequently $E_p = (|\vec{p}|^2 + m^2_{phys})^{1/2}$ with the physical mass. As a result $\phi_{in}(x)$ verifies the Klein-Gordon equation:

$$\boxed{(\Box + m^2_{phys})\phi_{in}(x) = 0}$$

Clearly, the difficult part is the exact definition of the limit that relates $\phi(x)$ to $\phi_{in}(x)$.

i) The simplest choice would be: $\boxed{\lim_{t\to-\infty}\phi(x) \overset{?}{=} \phi_{in}(x)}$

This is incorrect because we cannot never turn off self interactions

ii) The second possibility is to write: $\lim_{t\to-\infty}\phi(x) \overset{?}{=} \sqrt{c_{in}}\,\phi_{in}(x)$

This is also incorrect. The reason is that the Heisemberg fields are quantized by imposing the equal-time commutation relations: $[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$ and the point is that also the "in" fields satisfy the same relation: $[\phi_{in}(\vec{x},t),\dot{\phi}(y,t)] = i\delta(\vec{x}-\vec{y})$. Therefore if the limit is understood in terms of operators, it would imply $c=1$.

iii) One possibility is to interpret the previous limit in the sense of weak convergence i.e. the limit holds only at the level of matrix elements of operators (while it doesn't hold for operators). We write:

$$\boxed{\lim_{t\to-\infty}\phi(x) = \sqrt{c_{in}}\,\phi_{in}(x) \quad \text{in the weak sense}}$$

The value of $\sqrt{c_{in}}$ is interesting to have the following considerations:

1) $\phi(x)$ is an interacting field and it obeys: $e^{ia_\mu P^\mu}\phi(x)e^{-ia_\mu P^\mu} = \phi(x+a) \xrightarrow{\substack{a^\mu=-x^\mu \\ e^{-ix_\mu P^\mu}\phi(x)e^{ix_\mu P^\mu}=\phi(0)}}$ $\boxed{e^{ix_\mu P^\mu}\phi(0)\,e^{-ix_\mu P^\mu}=\phi(x)}$

2) Consider the matrix element $\langle\Omega|\phi(x)|\vec{p}\rangle_{in}$. We write:

$$\langle\Omega|\phi(x)|\vec{p}\rangle_{in} = \underbrace{\langle\Omega|e^{ix_\mu P^\mu}}_{\text{invariant of the vacuum}}\phi(0)\underbrace{e^{-ix_\mu P^\mu}|\vec{p}\rangle_{in}}_{|\vec{p}\rangle_{in}\text{ is an eigenstate of }P^\mu} = \langle\Omega|\phi(0)e^{-ix\cdot p}|\vec{p}\rangle_{in} = e^{-ix\cdot p}\langle\Omega|\phi(0)|\vec{p}\rangle_{in}$$

$$\longrightarrow \boxed{\langle\Omega|\phi(x)|\vec{p}\rangle_{in} = e^{-ix\cdot p}\langle\Omega|\phi(0)|\vec{p}\rangle_{in}}$$

3) We apply the Klein-Gordon operator:

$$(\Box + m^2_{phys})\langle\Omega|\phi(x)|\vec{p}\rangle_{in} = (\Box + m^2_{phys})\langle\Omega|\phi(0)|\vec{p}\rangle_{in}\,e^{-ip\cdot x} = 0 \quad \text{Since } (\Box + m^2_{phys})e^{-ipx} = (-p^2 + m^2_{phys})e^{-ip\cdot x} = 0$$

So the quantity $\langle\Omega|\phi(x)|\vec{p}\rangle_{in}$ verifies the K.G. equation even though $\phi(x)$ does not.

4) We consider the same quantity but now for $\phi_{in}(x)$, $\langle\Omega|\phi_{in}(x)|\vec{p}\rangle_{in}$. $\phi_{in}(x)$ admits a mode expansion so we compute:

$$\langle\Omega|\int\frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_k}}\left[a_{in}(\vec{k})e^{-ik\cdot x} + a^\dagger_{in}(\vec{k})e^{ik\cdot x}\right]\sqrt{2E_p}\,a^\dagger_{in}(\vec{p})|\Omega\rangle = \langle\Omega|\int\frac{d^3\vec{k}\sqrt{2E_p}}{(2\pi)^3\sqrt{2E_k}}e^{-ik\cdot x}(2\pi)^3\delta(\vec{k}-\vec{p})|\Omega\rangle = e^{-ip\cdot x}$$

$$\longrightarrow \boxed{\langle\Omega|\phi_{in}(x)|\vec{p}\rangle_{in} = e^{-ip\cdot x}}$$

5) We consider the quantity:

$$\boxed{A(t) = \int d^3\vec{x}\, e^{ip\cdot x}\overleftrightarrow{\partial_t}\langle\Omega|\phi(x)|\vec{p}\rangle_{in}} \qquad \overleftrightarrow{\partial_t} = \overrightarrow{\partial_t} - \overleftarrow{\partial_t}$$

this quantity does not depend on time. We can compute indeed explicitily the derivatives.

$$A(t) = \int d^3\vec{x} \left[e^{-ip\cdot x} \left(\partial_t \langle \Omega | \phi(x) | \vec{p} \rangle_{in} \right) - \left(\partial_t e^{ipx} \right) \langle \Omega | \phi(x) | \vec{p} \rangle_{in} \right]$$

$$\frac{dA(t)}{dt} = \int d^3\vec{x} \left[\cancel{\left(\partial_t e^{ip\cdot x}\right)\left(\partial_t \langle \Omega | \phi(x) | \vec{p} \rangle_{in}\right)} + e^{ipx} \partial_t^2 \langle \Omega | \phi(x) | \vec{p} \rangle_{in} - \left(\partial_t^2 e^{ipx}\right) \langle \Omega | \phi(x) | \vec{p} \rangle_{in} - \cancel{\left(\partial_t e^{ipx}\right)\left(\partial_t \langle \Omega | \phi(x) | \vec{p} \rangle_{in}\right)} \right.$$

The Key point is that both e^{ipx} and $\langle \Omega | \phi(x) | \vec{p} \rangle_{in}$ verify the KG equation. Therefore:

$$\begin{cases} \partial_t^2 e^{ipx} = \vec{\nabla}^2 e^{ipx} - m_{phys}^2 e^{ipx} \\ \partial_t^2 \langle \Omega | \phi(x) | \vec{p} \rangle_{in} = \vec{\nabla}^2 \langle \Omega | \phi(x) | \vec{p} \rangle_{in} - m_{phys}^2 \langle \Omega | \phi(x) | \vec{p} \rangle_{in} \end{cases}$$

$$\longrightarrow \frac{dA}{dt} = \int d^3\vec{x} \left[e^{ipx} \vec{\nabla}^2 \langle \Omega | \phi(x) | \vec{p} \rangle_{in} - \cancel{m_{phys}^2 e^{ipx} \langle \Omega | \phi(x) | \vec{p} \rangle_{in}} + \right.$$
$$\left. - \left(\vec{\nabla}^2 e^{ipx}\right) \langle \Omega | \phi(x) | \vec{p} \rangle_{in} + \cancel{m_{phys}^2 e^{ip\cdot x} \langle \Omega | \phi(x) | \vec{p} \rangle_{in}} \right]$$

Integrating by parts and using that boundary terms at spatial infinity vanish, we get:

$$= \int d^3\vec{x} \left[-\left(\vec{\nabla} e^{ip\cdot x}\right) \cdot \left(\vec{\nabla} \langle \Omega | \phi(x) | \vec{p} \rangle_{in}\right) + \left(\vec{\nabla} e^{ip\cdot x}\right) \cdot \left(\vec{\nabla} \langle \Omega | \phi(x) | \vec{p} \rangle_{in}\right) \right] = 0$$

We now elaborate the results obtained so far as follows:

$$\int d^3\vec{x}\, e^{ipx} \overleftrightarrow{\partial_t} \langle \Omega | \phi(x) | \vec{p} \rangle_{in} = \langle \Omega | \phi(0) | \vec{p} \rangle_{in} \int d^3\vec{x}\, e^{ipx} \overleftrightarrow{\partial_t} e^{-ipx}$$

this quantity is also independent on time since e^{-ipx} verifies KG. eq.

Since that equation does depend on time we can just evaluate it at $t \to -\infty$.
Since $\langle \Omega | \phi(x) | \vec{p} \rangle_{in}$ is a matrix element we use the weak convergence:

$$\int d^3\vec{x}\, e^{ip\cdot x} \overleftrightarrow{\partial_t} \sqrt{C_{in}} \langle \Omega | \phi_{in}(x) | \vec{p} \rangle_{in} = \langle \Omega | \phi(0) | \vec{p} \rangle_{in} \int d^3\vec{x}\, e^{ipx} \overleftrightarrow{\partial_t} e^{-ip\cdot x}$$

From the comparison (we use $\langle \Omega | \phi_{in}(x) | \vec{p} \rangle_{in} = e^{-ipx}$) we get:

$$\boxed{\langle \Omega | \phi(0) | \vec{p} \rangle_{in} = \sqrt{C_{in}}}$$

Consequently from point 2) we arrive at:

$$\langle \Omega | \phi(x) | \vec{p} \rangle_{in} = e^{-ixp} \langle \Omega | \phi(0) | \vec{p} \rangle_{in} = e^{-ixp} \sqrt{C_{in}} \overset{\text{using 4)}}{=} \sqrt{C_{in}} \langle \Omega | \phi_{in}(x) | \vec{p} \rangle_{in}$$

$$\longrightarrow \boxed{\langle \Omega | \phi(x) | \vec{p} \rangle_{in} = \sqrt{C_{in}} \langle \Omega | \phi_{in}(x) | \vec{p} \rangle_{in} = \sqrt{C_{in}}\, e^{-ix\cdot p}}$$

One can run the same arguments in the case of "out states". In short, we introduce the "out" field $\phi_{out}(x)$ with the weak convergence condition:

$$\boxed{\lim_{t \to +\infty} \phi(x) = \sqrt{C_{out}}\, \phi_{out}(x) \quad \text{in the Weak sense}}$$

where:

$$\phi_{out}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_p}} \left[a_{out}(\vec{p}) e^{-ipx} + a_{out}^{+}(\vec{p}) e^{ipx} \right] \; ; \; |\vec{p}\rangle_{out} = \sqrt{2E_p}\, a_{out}^{+}(\vec{p}) |\Omega\rangle \; ; \; \left[a_{out}(\vec{p}), a_{out}^{+}(\vec{p}) \right] = (2\pi)^3 \delta(\vec{p} - \vec{K})$$

A completely analogue computation gives :

$$\boxed{\langle\Omega|\phi(x)|\vec{p}\rangle_{out} = \sqrt{C_{out}}\,\langle\Omega|\phi_{out}(x)|\vec{p}\rangle_{out} = \sqrt{C_{out}}\,e^{-ix\cdot p}}$$

An important consideration is the following. If we consider one particle states, then we have :

$$\boxed{|\vec{p}\rangle_{in} = |\vec{p}\rangle_{out} \equiv |\vec{p}\rangle} \qquad \text{(only one type of single particle state)}$$

Physically that is because there is no scattering with only one particle and we must have :

$$S_{fi} = {}_{out}\langle\vec{k}|\vec{p}\rangle_{in} \propto \delta(\vec{k}-\vec{p})$$

as a consequence we have :

$$\langle\Omega|\phi(x)|\vec{p}\rangle = \sqrt{C_{in}}\,e^{-ip\cdot x} = \sqrt{C_{out}}\,e^{-ipx} \longrightarrow C_{n} = C_{out}$$

so that we only have :

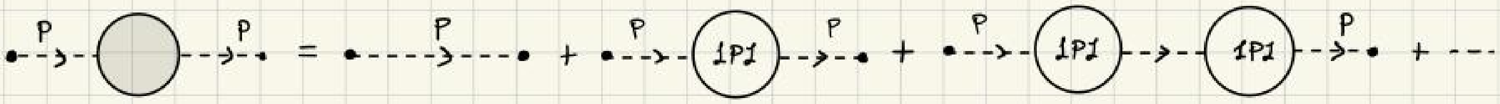
$$\boxed{\lim_{t\to\mp\infty}\phi(x) = \sqrt{C}\,\phi_{in/out}(x) \quad \text{in the weak sense} \quad \text{with} \quad \langle\Omega|\phi(x)|\vec{p}\rangle = \sqrt{C}\,\langle\Omega|\phi_{in/out}(x)|\vec{p}\rangle = \sqrt{C}\,e^{-ipx}}$$

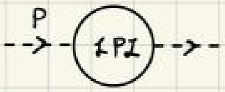
What is "c" ?

From the previous result, we see that : $\boxed{|\langle\Omega|\phi(x)|\vec{p}\rangle|^2 = c}$

So c can be interpreted as the probability to create a single-particle state from the vacuum. Bearing in mind this result, consider the following argument:

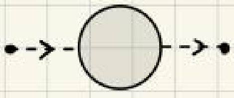
i) Consider the propagator in the *scalar theory*. *We can compute the dressed propagator, that is the propagator in the fully theory - by means of a simple resummation of diagrams similar in spirit to* the one we did for fermions and photons. Schematically:



where 

$\equiv i\Pi(p^2)$

The resummation gives :



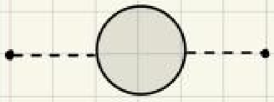
$$= \frac{i}{p^2-m^2+i\varepsilon} + \frac{i}{p^2-m^2+i\varepsilon}\,i\Pi(p^2)\,\frac{i}{p^2-m^2+i\varepsilon} +$$
$$+ \frac{i}{p^2-m^2+i\varepsilon}\,i\Pi(p^2)\,\frac{i}{p^2-m^2+i\varepsilon}\,i\Pi(p^2)\,\frac{i}{p^2-m^2+i\varepsilon} = \frac{i}{p^2-m^2+\Pi(p^2)+i\varepsilon}$$

Now, knowing the pole equation $m_{phys}^2 - m^2 + \Pi(p^2 = m^2_{phys}) = 0$ we can perform a Laurent expansion around the pole:

$$\frac{i}{p^2-m^2+\Pi(p^2)+i\varepsilon} \approx \frac{i}{p^2-m^2+\Pi(m^2_{phys}) + \left.\frac{d\Pi}{dp^2}\right|_{p^2=m^2_{phys}}(p^2-m^2_{phys}) + ---} = \frac{i}{(p^2-m^2_{phys})\left[1+\left.\frac{d\Pi}{dp^2}\right|_{p^2=m^2_{phys}} + O(p^2-m^2_{phys})\right.} = \frac{i\mathcal{Z}}{p^2-m^2_{phys}+i\varepsilon} + \text{reg terms}$$

where $$\mathcal{Z} \equiv \left(1+\left.\frac{d\Pi}{dp^2}\right|_{m_{phys}}\right)^{-1}$$

$\longrightarrow$



$$= \frac{iZ}{p^2-m^2_{phys}+i\varepsilon} + \text{terms regular in } p^2=m^2_{phys}$$

ii) We now try to interpret this result. We consider first the free theory. The propagator is:

$$\langle 0|T[\phi(x)\phi(y)]|0\rangle = \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

- consider the case $x^0>y^0$ and focus on $\langle 0|\phi(x)\phi(y)|0\rangle$. We introduce a resolution of the identity

$$\boxed{\langle 0|\phi(x)\,\mathbb{1}\,\phi(y)|0\rangle \quad ; \quad \mathbb{1} = \sum_X \int d\Pi_X\, |X\rangle\langle X|}$$

The resolution of the identity includes all states, single- and multiparticle states.

n.b. $d\Pi_X \equiv \prod_{i\in X} \frac{d^3\vec{p}_i}{(2\pi)^3(2E_{p_i})}$.

So we have that:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3(2E_p)} \langle 0|\phi(x)|\vec{p}\rangle\langle\vec{p}|\phi(y)|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3(2E_p)} e^{-ipx}e^{ipy}$$

so in this case is evident that we're propagating a single particle state from y to x.

- if we add the case $x^0<y^0$ we reconstruct the full propagator:

$$\langle 0|T[\phi(x)\phi(y)]|0\rangle = \theta(x^0-y^0)\int\frac{d^3\vec{p}}{(2\pi)^3 2E_p} \underbrace{\langle 0|\phi(x)|\vec{p}\rangle\langle\vec{p}|\phi(y)|0\rangle}_{e^{-ip(y-x)}} + \theta(y^0-x^0)\int\frac{d^3\vec{p}}{(2\pi)^3(2E_p)} \underbrace{\langle 0|\phi(y)|\vec{p}\rangle\langle\vec{p}|\phi(x)|0\rangle}_{e^{-ip(x-y)}}$$

$$= \int\frac{d^4\vec{p}}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2+i\varepsilon}$$

iii) Consider now the full propagator. We try the same trick, and insert a resolution of the identity.

$$\boxed{\langle \Omega|\underset{\substack{\downarrow\\ \text{interacting field}}}{\phi(x)}\phi(y)|\Omega\rangle = \sum_X \int d\Pi_X \underbrace{\langle\Omega|\phi(x)|X\rangle\langle X|\phi(y)|\Omega\rangle}}$$

the main difference is that now the interacting field has the capability of creating from $|\Omega\rangle$ not only single particle states, but also multiparticle states.

This is confirmed by the structure of the Dressed Propagator we found before. If we separate in the previous expression the one-particle states from the rest, we write:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int\frac{d^3\vec{p}}{(2\pi)^3(2E_p)}\langle\Omega|\phi(x)|\vec{p}\rangle\langle\vec{p}|\phi(y)|\Omega\rangle + \text{multiparticle intermediate states} =$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3(2E_p)} c\, e^{-ip(x-y)} + \text{multiparticle intermediate states}$$

↳ non zero in the interacting theory

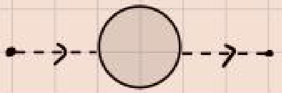
If we combine all in the propagator, we find:

$$\boxed{\langle\Omega|T[\phi(x)\phi(y)]|\Omega\rangle = c\underbrace{\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{p^2-m^2_{phys}+i\varepsilon}} + \text{multiparticle intermediate states}}$$

↳ in mom. space this is precisely the contribution of the 1st term of the Laurent expansion

We identify $c = Z$, and we thus have :

$$\lim_{t\to\mp\infty}\phi(x) = \sqrt{Z}\,\phi_{in/out}(x) \quad \text{(weak sense)}$$



$$= \frac{iZ}{p^2-m^2_{phys}+i\varepsilon} + \text{regular terms}$$

THE LSZ REDUCTION

We consider for simplicity the case of a 2 to 2 scattering

$$\boxed{{}_{out}\langle p_1,p_2|q_1,q_2\rangle_{in} = S_{fi}}$$

Step 1 :

$$S_{fi} = \sqrt{2E_{q_1}}\;{}_{out}\langle p_1,p_2|\,a^\dagger_{in}(\vec{q}_1)\,|q_2\rangle_{in} =$$

$$= \sqrt{2E_{q_1}}\;{}_{out}\langle p_1,p_2|(a^\dagger_{in}(\vec{q}_1)+a^\dagger_{out}(\vec{q}_1)-a^\dagger_{out}(\vec{q}_1))\,|q_2\rangle_{in} =$$

$$= \sqrt{2E_{q_1}}\;{}_{out}\langle p_1,p_2|\left[a^\dagger_{in}(\vec{q}_1)-a^\dagger_{out}(\vec{q}_1)\right]|q_2\rangle_{in} + \sqrt{2E_{q_1}}\;{}_{out}\underbrace{\langle p_1,p_2|\left[a^\dagger_{out}(\vec{q}_1)\right]}|q_2\rangle_{in}$$

acting on the left, the action of $a^\dagger_{out}(\vec{q}_1)$ gives zero unless either $\vec{p}_1=\vec{q}_1$ or $\vec{p}_2=\vec{q}_1$.

To keep the computation simple, we assume that all momenta are different, so to exclude cases in which some of the particles simply "run through" without partecipating to the scattering. Under this assumption :

$$\boxed{S_{fi} = \sqrt{2E_{q_1}}\;{}_{out}\langle p_1,p_2|\left[a^\dagger_{in}(\vec{q}_1)-a^\dagger_{out}(\vec{q}_1)\right]|q_2\rangle_{in}}$$

Step 2 : We invert in and out fields

Consider the "in" field: $\phi_{in} = \displaystyle\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[a_{in}(\vec{p})e^{-ip\cdot x}+a^\dagger_{in}(\vec{p})e^{ip\cdot x}\right]$

we compute:

$$\int d^3\vec{x}\,(\partial_t\phi_{in}(x))\,e^{ik\cdot x} = \int d^3\vec{x}\,e^{ikx}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[a_{in}(\vec{p})(-ip^0)e^{-ip\cdot x}+a^\dagger_{in}(\vec{p})(ip^0)e^{ip\cdot x}\right] =$$

$$= \int\frac{d^3\vec{x}\,d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[a_{in}(\vec{p})(-ip^0)e^{-i(p^0-k^0)t}e^{-i(\vec{k}-\vec{p})\vec{x}}+a^\dagger_{in}(\vec{p})(ip^0)e^{+i(p^0+k^0)t}e^{-i\vec{x}\cdot(\vec{k}+\vec{p})}\right] =$$

$$= \int\frac{d^3\vec{x}\,d^3\vec{p}}{\cancel{(2\pi)^3}\sqrt{2E_p}}\,E_p\left[a_{in}(\vec{p})(-i)e^{-i(p^0-k^0)t}\cancel{(2\pi)^3}\delta(\vec{k}-\vec{p}) + a^\dagger_{in}(\vec{p})(i)e^{+i(p^0+k^0)t}\cancel{(2\pi)^3}\delta(\vec{k}+\vec{p})\right]$$

The condition $|\vec{k}|=|\vec{p}|$ enforced by the delta function enforces $k^0=p^0$ if we take $k^2=(k^0)^2-|\vec{k}|^2=m^2_{phys}$

$$=\sqrt{\frac{E_k}{2}}\left[(-i)\,a_{in}(\vec{k})+i\,a^\dagger_{in}(\vec{k})\,e^{2iE_kt}\right]=$$

$$=\sqrt{\frac{E_k}{2}}(-i)\left[a_{in}(\vec{k})-a^\dagger_{in}(-\vec{k})\,e^{2iE_kt}\right]$$

$$\longrightarrow\ \boxed{\frac{i}{2E_k}\int d^3\vec{x}\,\left(\partial_t\phi_{in}(x)\right)e^{ik\cdot x}=\frac{1}{2}\left[a_{in}(\vec{k})-a^\dagger_{in}(-\vec{k})\,e^{2iE_kt}\right]}$$

We now consider:

$$\int d^3\vec{x}\,\phi_{in}(x)\left(\partial_t e^{ik\cdot x}\right)=\int d^3\vec{x}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left[a_{in}(\vec{p})e^{-ip\cdot x}+a^\dagger_{in}(\vec{p})e^{ip\cdot x}\right](ik^0)\,e^{ik\cdot x}=$$

$$=\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}(ik^0)\,d^3\vec{x}\left[a_{in}(\vec{p})e^{-ix(p-k)}+a^\dagger_{in}(\vec{p})e^{i(p+k)\cdot x}\right]=$$

$$=\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\,iE_k\,d^3\vec{x}\left[a_{in}(\vec{p})\,e^{-it(p^0-k^0)}e^{i\vec{x}\cdot(\vec{p}-\vec{k})}+a^\dagger_{in}(\vec{p})\,e^{it(p^0+k^0)}e^{-i\vec{x}\cdot(\vec{p}+\vec{k})}\right]=$$

$$=\int\frac{d^3\vec{p}}{\sqrt{2E_p}}\,E_k\cdot i\left[a_{in}(\vec{p})\,e^{-it(p^0-k^0)}\delta(\vec{p}-\vec{k})+a^\dagger_{in}(\vec{p})\,e^{it(p^0+k^0)}\delta(\vec{p}+\vec{k})\right]=$$

$$=i\sqrt{\frac{E_k}{2}}\left[a_{in}(\vec{k})+a^\dagger_{in}(-\vec{k})\,e^{2itE_k}\right]$$

$$\longrightarrow\ \boxed{\frac{i}{2E_k}\int d^3\vec{x}\,\left(\partial_t e^{ikx}\right)\phi_{in}(x)=-\frac{1}{2}\left[a_{in}(\vec{k})+a^\dagger_{in}(-\vec{k})\,e^{2itE_k}\right]}$$

If we take the difference, we find:

$$\frac{i}{2E_k}\int d^3\vec{x}\left[e^{ik\cdot x}\left(\partial_t\phi_{in}(x)\right)e^{ik\cdot x}-\left(\partial_t e^{ikx}\right)\phi_{in}(x)\right]\overset{!}{=}$$

$$=\frac{1}{2}\left[a_{in}(\vec{k})+\cancel{a^\dagger_{in}(-\vec{k})e^{2itE_k}}+a_{in}(\vec{k})-\cancel{a^\dagger_{in}(-\vec{k})e^{2iE_kt}}\right]=a_{in}(\vec{k})$$

$$\longrightarrow\ \boxed{\begin{aligned}a_{in}(\vec{k})&=\frac{i}{\sqrt{2E_k}}\int d^3\vec{x}\,e^{ik\cdot x}\overleftrightarrow{\partial_t}\,\phi_{in}(x)\\ a^\dagger_{in}(\vec{k})&=\frac{-i}{\sqrt{2E_k}}\int d^3\vec{x}\,e^{-ik\cdot x}\overleftrightarrow{\partial_t}\,\phi_{in}(x)\end{aligned}}$$

Consequently we write these formulas:

$$a^\dagger_{in}(\vec{q}_1)-a^\dagger_{out}(\vec{q}_1)=\frac{i}{\sqrt{2E_{q_1}}}\int d^3\vec{x}\left[\phi_{in}(x)\overleftrightarrow{\partial_t}\,e^{-iq_1\cdot x}-\phi_{out}(x)\overleftrightarrow{\partial_t}\,e^{-iq_1\cdot x}\right]=$$

$$=\frac{i}{\sqrt{2E_{q_1}}}\int d^3\vec{x}\left[\phi_{in}(x)-\phi_{out}(x)\right]\overleftrightarrow{\partial_t}\,e^{-iq_1\cdot x}$$

$$\longrightarrow\ \boxed{a^\dagger_{in}(\vec{q}_1)-a^\dagger_{out}(\vec{q}_1)=\frac{i}{\sqrt{2E_{q_1}}}\int d^3\vec{x}\left[\phi_{in}(x)-\phi_{out}(x)\right]\overleftrightarrow{\partial_t}\,e^{-iq_1\cdot x}}$$

Step 3: we notice that the above expression does not depend on time. Consequently it can be evaluated at any chosen point of time $x^0 \equiv t$.

In particular we consider the 1st term at $x^0 \longrightarrow -\infty$ and the second term at $x^0 \longrightarrow +\infty$ so that we can use the weak limit to write ϕ_{in} and ϕ_{out} in terms of $\phi(x)$

$$\boxed{a^\dagger_{in}(\vec{q}_1) - a^\dagger_{out}(\vec{q}_1) = \frac{i}{\sqrt{2E_{q_1}}} \int \frac{d^3\vec{x}}{\sqrt{Z}} \left[\lim_{t \to -\infty} - \lim_{t \to +\infty}\right] \phi(x) \overleftrightarrow{\partial}_t e^{-iq_1 \cdot x}}$$

Step 4: we switch to a 4-dim integral

We use the identity: $\displaystyle\int_{-\infty}^{+\infty} dx^0 \frac{\partial}{\partial x^0} F(x) = \left(\lim_{x \to +\infty} - \lim_{x \to -\infty}\right) F(x)$, that gives:

$$\begin{aligned}
a^\dagger_{in}(\vec{q}_1) - a_{out}(\vec{q}_1) &= \frac{(-i)}{\sqrt{Z}} \frac{1}{\sqrt{2E_{q_1}}} \int d^4\vec{x} \frac{\partial}{\partial x^0}\left[\phi(x) \overleftrightarrow{\partial}_{x^0} e^{-iq_1 \cdot x}\right] = \\
&= \frac{(-i)}{\sqrt{Z}\sqrt{2E_{q_1}}} \int d^4x \frac{\partial}{\partial x^0}\left[\phi(x)(\partial_{x^0} e^{-iq_1 \cdot x}) - (\partial_{x^0}\phi(x)) e^{-iq_1 \cdot x}\right] = \\
&= \frac{(-i)}{\sqrt{Z}\sqrt{2E_{q_1}}} \int d^4x \left[\phi(x)\underbrace{(\partial^2_{x^0} e^{-iq_1 \cdot x})}_{KG\ eq.:\ (\Box + m^2_{phys})e^{-iq_1 \cdot x} = 0\ \longrightarrow\ \partial^2_{x^0} e^{-iq_1 \cdot x} = (\vec{\nabla}^2 - m^2_{phys}) e^{-iq_1 x}} - (\partial^2_{x^0}\phi(x)) e^{-iq_1 \cdot x}\right] = \\
&= \frac{(-i)}{\sqrt{Z}\sqrt{2E_{q_1}}} \int d^4x \left[\phi(x)(\vec{\nabla}^2 - m^2_{phys}) e^{-iq_1 \cdot x} - (\partial^2_{x^0}\phi(x)) e^{-iq_1 \cdot x}\right] = \\
&= \frac{(-i)}{\sqrt{Z}\sqrt{2E_{q_1}}} \int d^4x \left[(\vec{\nabla}^2 - m^2_{phys} - \partial^2_{x^0})\phi(x)\right] e^{-iq_1 \cdot x} \\
&= \frac{(+i)}{\sqrt{Z}\sqrt{2E_{q_1}}} \int d^4x \left[(\Box_x + m^2_{phys})\phi(x)\right] e^{-iq_1 \cdot x}
\end{aligned}$$

$$\boxed{a^\dagger_{in}(\vec{q}_1) - a_{out}(\vec{q}_1) = \frac{(+i)}{\sqrt{Z}\sqrt{2E_{q_1}}} \int d^4x \left[(\Box_x + m^2_{phys})\phi(x)\right] e^{-iq_1 \cdot x}}$$

Step 5: we go back to the matrix element

$$\boxed{S_{fi} = \frac{i}{\sqrt{Z}} \int d^4x \ {}_{out}\langle p_1 p_2 | \phi(x) | q_2 \rangle_{in} \left(\overleftarrow{\Box}_x + m^2_{phys}\right) e^{-iq_1 \cdot x}}$$

Step 6: the above procedure can be generalized to the other particles

As an exercise consider the extraction of the 2nd particle from the initial state.

$${}_{out}\langle p_1, p_2 | \phi(x) | q_2 \rangle_{in} = \sqrt{2E_{q_2}} \ {}_{out}\langle p_1, p_2 | \phi(x) a^\dagger_{in}(\vec{q}_2) | \Omega \rangle$$

$$\begin{aligned}
\phi(x) a^\dagger_{in}(\vec{q}_2) &= \phi(x) \frac{i}{\sqrt{2E_{q_2}}} \int d^3\vec{y} \ \phi_{in}(y) \overleftrightarrow{\partial}_{y_0} e^{-iq_2 y} \\
&= \frac{i}{\sqrt{Z}\sqrt{2E_{q_2}}} \lim_{y^0 \to -\infty} \int d^3\vec{y} \ \phi(x)\phi(y) \overleftrightarrow{\partial}_{y_0} e^{-iq_2 \cdot y} \\
&= \frac{i}{\sqrt{Z}\sqrt{2E_{q_2}}} \lim_{y^0 \to -\infty} \int d^3\vec{y} \ T[\phi(x)\phi(y)] \overleftrightarrow{\partial}_{y_0} e^{-iq_2 \cdot y}
\end{aligned}$$

we can introduce Time Ordering without problems because of the $\lim y^0 \to -\infty$

$$= \frac{i}{\sqrt{Z}\sqrt{2E_{q_2}}} \lim_{y^0 \to +\infty} \int d^3\vec{y}\, T[\phi(x)\phi(y)] \overleftrightarrow{\partial}_{y_0} e^{-iq_2\cdot y} - \frac{i}{\sqrt{Z}\sqrt{2E_{q_2}}} \int d^4y\, \partial_{y_0}\left\{T[\phi(x)\phi(y)] \overleftrightarrow{\partial}_{y_0} e^{-iq_2\cdot y}\right\}$$

Consider the 1st term, since there is a lim $y^0 \to +\infty$ we write:

$$\frac{i}{\sqrt{Z}\sqrt{2E_{q_2}}} \lim_{y^0 \to +\infty} \int d^3\vec{y}\, \phi(y) \overleftrightarrow{\partial}_{y_0} e^{-iq_2\cdot y} \phi(x) = \frac{i}{\sqrt{2E_{q_2}}} \int d^3\vec{y}\, \phi_{out}(y) \overleftrightarrow{\partial}_{y_0} e^{-iq_2\cdot y} \phi(x) = a^\dagger_{out}(\vec{q}_2)\phi(x)$$

and it vanishes since we assume that all p_i are different from the q_i.
The 2nd term can be analysed as we did before. We thus find:

$$\phi(x)\, a^\dagger_{in}(\vec{q}_2) = \frac{i}{\sqrt{Z}} \int d^4y\, T[\phi(x)\phi(y)] (\overleftarrow{\Box}_y + m^2_{phys})\, e^{-iq_2 y}$$

so that we write:

$$\boxed{S_{fi} = \left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4x\, d^4y\ {}_{out}\langle p_1, p_2 | T[\phi(x)\phi(y)] | \Omega\rangle (\overleftarrow{\Box}_x + m^2_{phys})(\overleftarrow{\Box}_y + m^2_{phys}) e^{-iq_1\cdot x} e^{-iq_2 y}}$$

This construction can be completed considering also the final states. We obtain the LSZ reduction formula for scalars:

$$\boxed{\begin{aligned} &{}_{out}\langle p_1, \dots, p_n | q_1, \dots, q_m \rangle_{in} = \text{disconnected terms} + \\ &+ \left(\frac{i}{\sqrt{Z}}\right)^{n+m} \int d^4y_1 \dots d^4y_n \int d^4x_1 \dots d^4x_m\, e^{i\left(\sum_{k=1}^{n} p_k\cdot y_k - \sum_{k=1}^{m} q_k\cdot x_k\right)} (\Box_{x_1} + m^2_{phys}) \dots (\Box_{y_1} + m^2_{phys}) \dots \langle\Omega| T[\phi(y_1)\dots\phi(y_n)\phi(x_1)\dots\phi(x_m)]|\Omega\rangle \end{aligned}}$$

This equation becomes even more evocative when written in momentum space. To this end we introduce the N-point Green function:

$$\boxed{G^{(N)}(p_1, \dots, p_N) \equiv \int \prod_{k=1}^{N} d^4x_k\, e^{ip_k\cdot x_k} \langle\Omega| T[\phi(x_1)\dots\phi(x_N)]|\Omega\rangle}$$

We integrate by parts so that the K.G. operators act on the plane waves factors; schematically it amounts to the substitution:

$$(\Box_x + m^2_{phys}) \longrightarrow (-q^2 + m^2_{phys})$$

We obtain the LSZ reduction formula in momentum space:

$$\boxed{S_{fi} = \left(\frac{-i}{\sqrt{Z}}\right)^{n+m} \lim_{q_1^2 \to m^2_{phys}} (q_1^2 - m^2_{phys}) \dots \lim_{p_n^2 \to m^2_{phys}} (p_n^2 - m^2_{phys})\ G^{(n+m)}(p_1, \dots, p_n; -q_1, \dots, -q_m)}$$

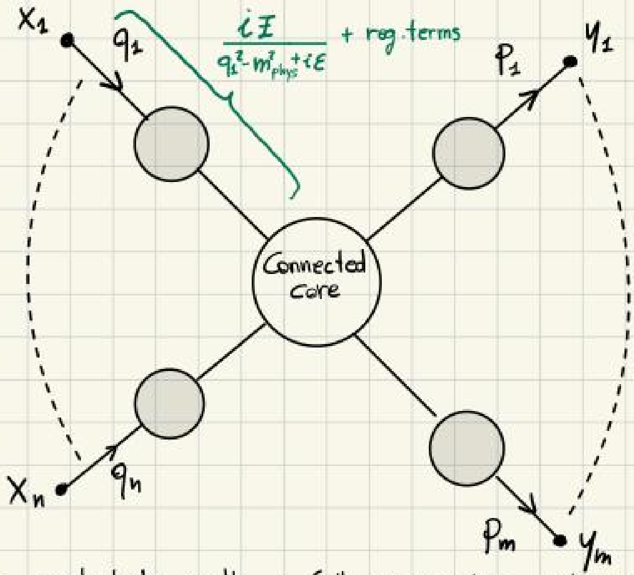
Notice that the ext. particles are physical and they are on the mass-shell. Therefore $q_i^2 = m^2_{phys}$ and $p_j^2 = m^2_{phys}$. For this reason, we used the notation with the limit written explicitely.

DISCUSSION

We give a diagrammatical interpretation of the LSZ reduction formulas in momentum space.
$G^{(n+m)}(p_1, \dots, p_n; -q_1, \dots, -q_m)$ it corresponds to the sum over all connected diagrams with n+m external points.
Think about $G^{(n+m)}$ as the generalization of the propagator. $G^{(N)}(p_1, \dots, p_N)$ is the Fourier transform of $\langle\Omega| T[\phi(x_1)\dots\phi(x_N)]|\Omega\rangle$. So it generalizes the propagator that is the Fourier transform of $\langle\Omega| T[\phi(x_1)\phi(x_2)]|\Omega\rangle$. Therefore the propagator has 2 ext points

the N-point Green function has N ext. points.

Diagrammatically we have the following structure for the connected N-point Green f.



Each external leg contribute with a full propagator, whose structure, if we Laurent expand around the single particle pole, is given by:

$$\frac{iZ}{p^2 - m^2_{phys} + i\varepsilon} + \text{reg. terms}$$

The key point is that when we compute the scattering matrix elements with on-shell external particles the LSZ formula tells us that S_{fi} will vanish, because of the factors $\lim_{q_i^2 \to m^2_{phys}} (q_i^2 - m^2_{phys})$.
Unless these factors are precisely canceled by simple poles in the Green's function.

Therefore the only non-vanishing contributions comes from the limit in which the external propagators are close to their single-particle poles. We thus find:

$$\boxed{S_{fi} = (\sqrt{Z})^{n+m} G^{(n+m)}_{AMP}(p_1, \dots, p_n; -q_1, \dots, -q_m)}$$

these are amputated Green functions with on-shell momenta $p_i^2 = m^2_{phys}$; $q_i^2 = m^2_{phys}$

In general we discovered that we need to include a factor $\sqrt{Z}$ for each external leg. N.B. At the free level $Z = 1$ and we do not tipically include this rule. However at loop level it becames important.

COMMENT : We worked out the previous result in a scalar theory but it remains true in theories with spin. For spinning theories, in addition to $\sqrt{Z}$, each leg has a corresponding spinor or polarization vector.

For our purposes we highlight the following points:

1) Scattering matrix elements are invariant under field rescaling

Suppose we make a field rescaling $\boxed{\phi(x) \longrightarrow \tilde{\phi}(x) = \frac{1}{A}\phi(x)}$ there are 2 consequences:

i) At the level of the propagator, we find:

$$\frac{1}{A^2} \langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle = \langle \Omega | T[\tilde{\phi}(x)\tilde{\phi}(y)] | \Omega \rangle$$

ii) this generalizes for the case of a N-point correlator:

$$\langle\Omega| T[\tilde\Phi(x_1)\dots\tilde\Phi(x_N)]|\Omega\rangle = \frac{1}{A^N}\langle\Omega|T[\Phi(x_1)\dots\Phi(x_N)]|\Omega\rangle$$

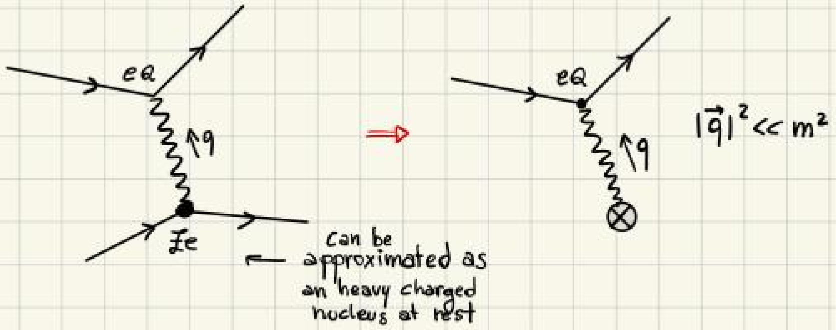
Therefore S_{fi} (take it in momentum space for simplicity) modifies in:

$$\tilde S_{fi} = \left(\frac{-iA}{\sqrt{Z}}\right)^{n+m} \cdot \lim_{q_i^2\to m^2_{phys}} (q_i^2 - m^2_{phys}) \dots \tilde G^{(n+m)}(p_1,\dots,p_n; q_1,\dots,q_m) =$$

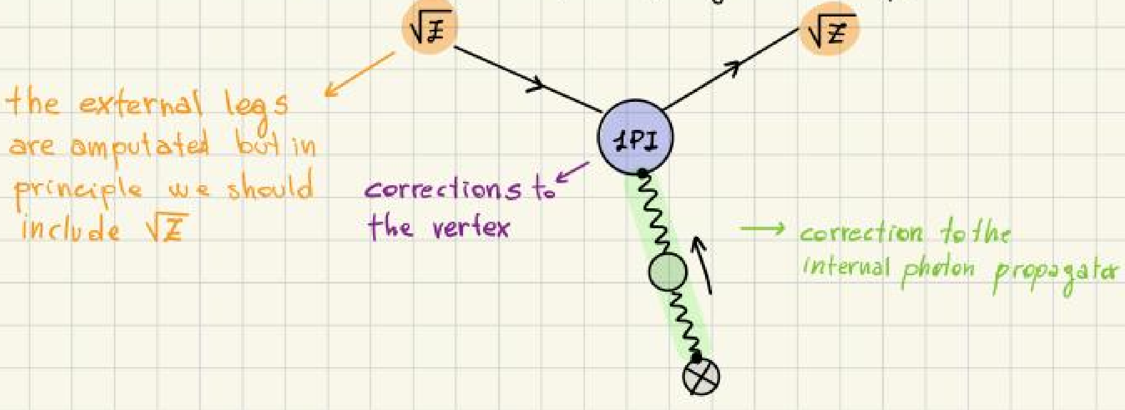
$$= \left(\frac{-i\cancel{A}}{\sqrt{Z}}\right)^{n+m} \cdot \lim_{q_i^2\to m^2_{phys}} (q_i^2 - m^2_{phys}) \dots \frac{1}{\cancel{A^{n+m}}} G^{(n+m)}(p_1,\dots,p_n; q_1,\dots,q_m) = S_{fi} \qquad \square$$

This is not a symmetry because $\mathcal{L}$ is changing so there is no current associated to this rescaling transformation. However on-shell matrix elements are invariant.

2) Consider the elastic scattering of charged particle in the non-relativistic limit. Working at the tree level we identified the parameter "e" of $\mathcal{L}$ as the fundamental electric charge



In reality one should consider the full interacting theory and the computation of the scattering matrix element takes the schematic form (diagrammatically):



Moral: in general "e" will not be equal to the physical electric charge.

FURRY THEOREM

Since the approach we constructed so far is general, all the identities ang propertie will remain valid at all order in pertubation theory.
Consider for example the N-points time ord. correlation function with N-photon fields.

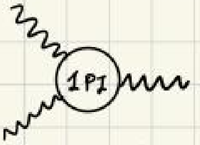
$$\langle\Omega|T[A^{\mu_1}(x_1)\dots A^{\mu_N}(x_N)]|\Omega\rangle$$

we apply $CC^{-1} = \mathbb{1}$:

$$\underbrace{\langle\Omega|C^{-1}}_{\text{vacuum inv.}} C\, T[A^{\mu_1}(x_1) \underbrace{C C^{-1} A^{\mu_2}(x_2) C}_{CA^\mu(x)C^{-1} = (-1)A^\mu(x)} C^{-1}\dots A^{\mu_N}(x_N)]C^{-1}C|\Omega\rangle = \langle\Omega|T[A^{\mu_1}(x_1)\dots A^{\mu_N}(x_N)]|\Omega\rangle(-1)^N$$

if N is odd then: $\boxed{\langle\Omega| T[A^{\mu_1}(x_1) \cdots A^{\mu_N}(x_N)]|\Omega\rangle = 0}$ Furry theorem

Furry theorem is the reason why we found:



@ 1 LOOP = 0

we now understand it is valid at all order in perturbation theory and for an odd number of external photon.

RENORMALIZED LAGRANGIAN AND COUNTERTERMS

Let's go back to QED:

$$\mathcal{L}_{QED} = \overline{\psi}_B (i\gamma^\mu \partial_\mu - m_B)\psi_B - \frac{1}{4} F_{\mu\nu_B} F^{\mu\nu}_B + e_B \overline{\psi}_B \gamma^\mu A_{\mu_B} \psi_B$$

This is our original lagrangian but we add a subscript "B" to m and e to remind that these are not the physical mass and the physical electric charge. We also added the same subscript to the fields. This is known as the Bare Lagrangian. We rewrite now it as follows:

1) We introduce the renormalized fields: $\boxed{\psi_B \equiv \sqrt{Z_2}\,\psi_R \;;\; A^\mu_B \equiv \sqrt{Z_3}\,A^\mu_R}$

We can do it since we have the freedom to rescale the fields without changing the physics. (that is without changing the on-shell matrix elements).

$$\longrightarrow \quad \mathcal{L}_B = -\frac{Z_3}{4}(\partial^\mu A^\nu_R - \partial^\nu A^\mu_R)^2 + Z_2 \overline{\psi}_R (i\not{\partial} - m_B)\psi_R + e_B Z_2 \sqrt{Z_3}\, \overline{\psi}_R \gamma_\mu A^\mu_R \psi_R$$

2) We introduce the renormalized mass and coupling: $\boxed{m_B \equiv Z_m m_R \;;\; e_B \equiv Z_e e_R}$

$$\longrightarrow \quad \mathcal{L}_B = -\frac{Z_3}{4}(\partial^\mu A^\nu_R - \partial^\nu A^\mu_R)^2 + Z_2 \overline{\psi}_R i\not{\partial}\psi_R + Z_2\sqrt{Z_3}\, Z_e e_R \overline{\psi}_R \not{A}_R \psi_R - Z_2 Z_m m_R \overline{\psi}_R \psi_R$$

From now on let me drop the "R" from renormalized fields for simplicity. Moreover we could define $\boxed{Z_1 \equiv Z_2 \sqrt{Z_3}\, Z_e}$. We arrive at:

$$\boxed{\mathcal{L}_B = -\frac{Z_3}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)^2 + i Z_2 \overline{\psi}\not{\partial}\psi - Z_2 Z_m m_R \overline{\psi}\psi + e_R Z \overline{\psi}\not{A}\psi}$$

3) Z_1, Z_2, Z_3 and Z_m are renormalization constants. It is convenient to introduce the redefinitions:

$$\boxed{\begin{aligned} Z_1 &= 1 + \delta_1 \\ Z_2 &= 1 + \delta_2 \qquad\qquad Z_2 Z_m m_R = m_R + \delta m \\ Z_3 &= 1 + \delta_3 \end{aligned}}$$

$\delta_1, \delta_2, \delta_3, \delta m$ are called counter terms. Substituting we see that:

$$\mathcal{L}_B = -\frac{(1+\delta_3)}{4} F_{\mu\nu}F^{\mu\nu} + i\,(1+\delta_2)\overline{\psi}\not{\partial}\psi - (m_R + \delta m)\overline{\psi}\psi + e_R(1+\delta_1)\overline{\psi}\not{A}\psi$$

We can separate the Lagrangian in 2 pieces:

$$\boxed{\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{CT}}$$

$$\boxed{\begin{aligned}\mathcal{L}_R &\equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\partial\!\!\!/\psi - m_R\overline{\psi}\psi + e_R\overline{\psi}A\!\!\!/\psi \\ \mathcal{L}_{CT} &\equiv -\frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\delta_2\partial\!\!\!/ - \delta m)\psi + \delta_1 e_R\overline{\psi}A\!\!\!/\psi\end{aligned}}$$

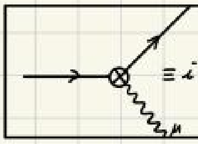
Comments :

We're not adding anything to the Bare Lagrangian ; it is an equality ; we just splitted the Bare Lagrangian in 2 parts. We expressed the freedom of the original Lagrangian in terms of these counter terms. (free parameters that we're going to use to cancel out our divergences.

$\mathcal{L}_R$ has the same structure of $\mathcal{L}_B$, but instead of Bare parameters and fields we have renormalized parameters and fields. What about $\mathcal{L}_{CT}$? What is its interpretation?

i) We start from $\boxed{\delta_1 e_R\overline{\psi}\gamma^\mu A_\mu\psi}$

This is clearly a new interaction term with Feynman rule :



$\equiv i\delta_1 e_R\gamma^\mu$

ii) Consider $\boxed{-\frac{\delta_3}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)}$

This is quadratic in the field , technically this is not an interaction . However it is useful to think about it as a "two-field" interaction term .

We rewrite: $-\frac{\delta_3}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{\delta_3}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu - \partial^\nu A^\mu) =$

$= +\frac{\delta_3}{2}A_\nu\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{\delta_3}{2}A_\nu(\partial_\mu\partial^\mu g^{\nu\rho}A_\rho - \partial_\rho\partial^\nu A^\rho) = \frac{\delta_3}{2}A_\nu(g^{\nu\rho}\partial_\mu\partial^\mu - \partial^\rho\partial^\nu)A_\rho$

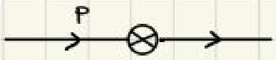
We introduce a Feynman rule to describe this "interaction" :

$\equiv i\delta_3(\underbrace{-g^{\nu\rho}p^2 + p^\nu p^\rho}_{})$

each derivative gives $(-ip_\mu)$ for incoming and (ip_μ) for outcoming particle

iii) Consider $\boxed{\overline{\psi}(i\delta_2\partial\!\!\!/ - \delta m)\psi}$

This can be seen as a sort of new "interaction" whose Feynman rule is :



$\equiv i(\delta_2 p\!\!\!/ - \delta m)$

incoming fermion gives a $(-i)p_\mu$

The on-shell renormalization scheme

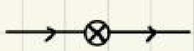
Consider the following example: consider the electron self energy at one loop

$\mathcal{L}_B \sim$ •

$= i\Sigma^{(B)}_{1LOOP}(p\!\!\!/) = iA^{(B)}(p^2) + ip\!\!\!/B^{(B)}(p^2)$

$\mathcal{L}_R + \mathcal{L}_{CT} \sim$ •

$+$

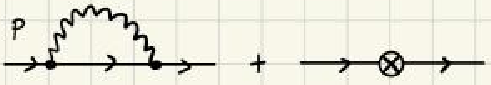


$= i\Sigma^{(R)}_{1LOOP}(p\!\!\!/) + i\Sigma^{(CT)}_{1LOOP}(p\!\!\!/)$

If we do the computation with $\mathcal{L}_R + \mathcal{L}_{CT}$, we now have two contributions :

- $\boxed{i\,\Sigma^{(R)}_{1LOOP}(\not p) \text{ computed with } \mathcal{L}_R}$: it has the same structure of $i\,\Sigma^{(B)}_{1loop}(\not p)$ with $m_B \to m_R$ and $e_B \to e_R$. Consequently, $i\,\Sigma^{(R)}_{1loop}(\not p)$ will diverge precisely as $i\,\Sigma^{(B)}_{1loop}(\not p)$ does
- $\boxed{i\,\Sigma^{(CT)}_{1LOOP}(\not p) \text{ computed wit } \mathcal{L}_{CT}}$: in this case we have the counter terms that we can choose to cancel the divergences.

We have the following result (focusing only on divergent terms).



$$= \left[\frac{i e_R^2}{(4\pi)^2}\left(\frac{1}{\varepsilon}\not p - \frac{4}{3}m_R\right) + \text{finite terms}\right] + i\,(\delta_2 \not p - \delta_m)$$

We consider the choice:

$$\boxed{\delta_2 = -\frac{e_R^2}{(4\pi)^2}\frac{1}{\varepsilon} + C_2} \qquad \boxed{\delta_m = \frac{-4e_R^2}{(4\pi)^2}\frac{m_R}{\varepsilon} + C_m}$$

it cancels the divergent part proportional to $\not p$

it cancels the divergent part not proportional to $\not p$

C_2 and C_m are arbitrarly. If we only care about the cancellation of UV divergences the choice of counterterms is not unique

In order to fix also the finite parts of the Counterterms we need to impose some conditions. Any prescription for choosing the finite parts of the Counterterms is Known as subtraction scheme. A subtraction scheme is specified by a set of conditions Known as renormalization conditions.

We consider the so called on-shell subtraction scheme. The renormalization conditions in the "os" scheme are :

$$\boxed{m_R \equiv m_{phys} \;\; ; \;\; e_R \equiv e_{phys}}$$

We need however 2 more conditions (the counter terms are four):

We require that both the electron and photon propagators have residue $Z=1$ at their pole

THE RENORMALIZED QED (AT ONE LOOP)

i) The electron self energy

We write : $\boxed{\Sigma_{1LOOP}(\not p) = \Sigma^{(R)}_{1LOOP}(\not p) + \delta_2\not p - \delta_m}$ ★

- The pole equation tells me that : $\underbrace{\cancel{m_{phys} - m_R}}_{\text{(on shell scheme)}} + \Sigma(\not p = m_{phys}) = 0 \longrightarrow \boxed{\Sigma(\not p = m_{phys}) = 0}$
- Let's impose the other condition:

$$Z = \left(1 + \left.\frac{d\Sigma}{d\not p}\right|_{\not p = m_{phys}}\right) \overset{!}{=} 1 \longrightarrow \boxed{\left.\frac{d\Sigma}{d\not p}\right|_{\not p = m_{phys}} = 0}$$

▸ Therefore from the derivative of ★ we find:

$$\underset{\shortparallel \atop 0}{\cancel{\frac{d\Sigma}{d\not p}}}\Bigg|_{\not p = m_{phys}} = \left.\frac{d\Sigma^{(R)}}{d\not p}\right|_{\not p = m_{phys}} + \delta_2 \longrightarrow \boxed{\delta_2 = -\left.\frac{d\Sigma^{(R)}_{1LOOP}}{d\not p}\right|_{\not p = m_{phys}}}$$

▸ While computing ★ at $\not{p} = m_{phys}$ we find:

$$\cancel{\Sigma_{1LOOP}(\not{p}=m_{phys})} = \Sigma^{(R)}_{1LOOP}(\not{p}=m_{phys}) + \delta_2 \cdot m_{phys} - \delta m \longrightarrow \boxed{\delta_m = \Sigma^{(R)}_{1LOOP}(\not{p}=m_{phys}) - m_{phys} \frac{d\Sigma^{(R)}_{1LOOP}}{d\not{p}}\Big|_{\not{p}=m_{phys}}}$$

$$\overset{\shortparallel}{0}$$

This choice of δ_2 and δ_m cancels the UV-divergences. We define :

$$\boxed{\overline{\Sigma}_{1LOOP}(\not{p}) \equiv \Sigma^{(R)}_{1LOOP}(\not{p}) + \delta_2 \cdot \not{p} - \delta_m}$$

Let's check that it's free from uvg diver ence:

$$\overline{\Sigma}_{1LOOP}(\not{p}) = \Sigma^{(R)}_{1LOOP}(\not{p}) - \frac{d\Sigma^{(R)}_{1LOOP}}{d\not{p}}\Big|_{\not{p}=m_{phys}} \cdot \not{p} - \Sigma^{(R)}_{1LOOP}(\not{p}=m_{phys}) + \frac{d\Sigma^{(R)}_{1LOOP}}{d\not{p}}\Big|_{\not{p}=m_{phys}} \cdot m_{phys} =$$

$$= \cancel{\Sigma^{(R)}_{1LOOP}(\not{p}=m_{phys})} + \cancel{\frac{d\Sigma^{(R)}_{1LOOP}}{d\not{p}}\Big|_{\not{p}=m_{phys}}(\not{p}-m_{phys})} + \text{finite terms} - \left[\cancel{\Sigma^{(R)}_{1LOOP}(\not{p}=m_{phys})} + \cancel{\frac{d\Sigma^{(R)}_{1LOOP}}{d\not{p}}\Big|_{\not{p}=m_{phys}}(\not{p}-m_{phys})}\right] =$$

$= \text{UV finite terms}$

$$\longrightarrow \boxed{\overline{\Sigma}_{1LOOP}(\not{p}) \text{ is UV-finite and it is now written in terms of physical parameters}}$$

ii) The photon self energy

We write : $$\boxed{i\,\Pi^{\mu\nu}_{1LOOP}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\, i\,\Pi^{(R)}_{1LOOP}(q) - i\delta_3 (g^{\mu\nu}q^2 - q^\mu q^\nu)}$$

The pole equation tells me $Z = (1 - \Pi(q^2=0))^{-1} \overset{!}{=} 1 \longrightarrow \boxed{\Pi(q^2=0) = 0}$

We impose this condition to fix δ_3 :

$$\Pi^{(R)}_{1LOOP}(q^2=0) - \delta_3 = 0 \longrightarrow \boxed{\delta_3 = \Pi^{(R)}_{1LOOP}(q^2=0}$$

We now define: $$\boxed{\overline{\Pi}_{1LOOP}(q^2) = i\,(q^2 g^{\mu\nu} - q^\mu q^\nu)\left(\Pi^{(R)}_{1LOOP}(q^2) - \Pi^{(R)}_{1LOOP}(q^2=0)\right)}$$

we already know that this difference does not diverge in the UV

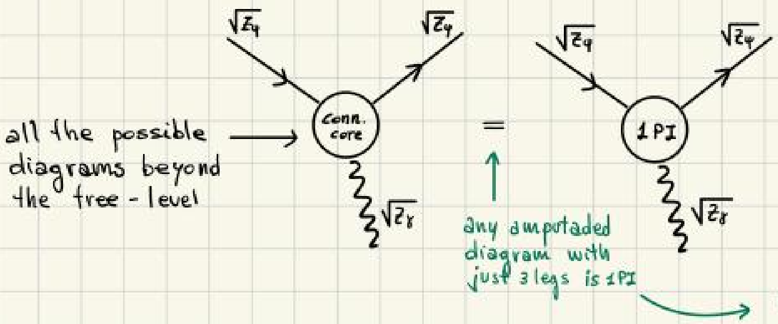
iii) Vertex correction

In the on-shell scheme, we impose the condition $e_R = e_{phys}$. Therefore, we need first of all a way to properly define the physical electric charge. Let us use as a guidance the tree-level intuition. We consider the amplitude for emission or absorbtion of a photon from an electron considering the soft photon limit $q \to 0$.

$$\lim_{q\to 0} i\mathcal{M}(q_1,q_2) \overset{!}{=} \bar{u}(q_2)\, i\, e_{phys}\, \gamma^\mu u(q_1)\, \varepsilon_\mu(q)$$

we can take it as a definition of the physical electric charge. In fact, if we consider the scattering of an electron off some heavy charged nucleus in the elastic, non relativistic limit (limit of low momentum transfer), the electron must interact with the current $\bar{u}(\bar{q}_2)\, i e_{phys}\, \gamma^\mu u(q_1)$ since e_{phys} is what we measure via Coulomb interaction.

At the tree level this definition correctly gives us $e = e_{phys}$. However in the fully interacting theory this definition has to be elaborated. The LSZ red. formula tells us that the amplitude for the emission/absorbtion of a photon takes the form:



in a 2 tot 2 scattering

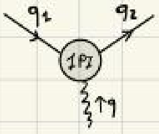
Since in the on-shell scheme we imposed $Z=1$ we arrive at:

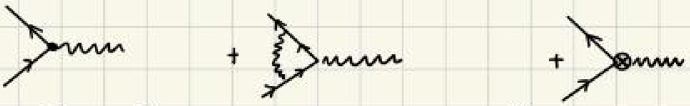
$$\lim_{q\to 0}$$

$$\overset{!}{=} \bar{u}(q_2)\, i e_{phys}\, \gamma^\mu u(q_1)\, \varepsilon_\mu(q)$$

We Know the explicit structure of this term, so we can set the equations:

We write





$$= i e_R\, \Gamma^\mu(q_1,q_2) = i e_R \left\{ \left[1 + F^{(R)}_{1,1loop}(q^2) \right] \gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m_R} q_\nu F_2(q^2) \right\} + i\,\delta_1 e_R \gamma^\mu$$

from $\mathcal{L}_R$ from $\mathcal{L}_{cr}$

We now compute the limit of $q \longrightarrow 0$

$$\longrightarrow \quad = i e_R \left[1 + F^{(R)}_{1,1loop}(q^2=0) \right] \gamma^\mu + i\,\delta_1 e_R \gamma^\mu \overset{!}{=} i e_{phys}\, \gamma^\mu$$

With the os condition $e_R = e_{phys}$ we et:

$$i e_{phys} \left[F^{(R)}_{1,1loop}(q^2=0) + \delta_1 \right] \gamma^\mu \overset{!}{=} i e_{phys}\, \gamma^\mu \longrightarrow \boxed{\delta_1 = -F^{(R)}_{1,1loop}(q^2=0)}$$

Finally we check UV-finiteness. We define:

$$\overline{\Gamma}^\mu_{1loop}(q_1,q_2) = \Gamma^{\mu\,(R)}_{1loop}(q_1,q_2) + \delta_1 \gamma^\mu$$

let's check:

$$\overline{\Gamma}^\mu_{1loop}(q_1,q_2) = \Gamma^{\mu\,(R)}_{1loop}(q_1,q_2) + \delta_1 \gamma^\mu = \Gamma^{\mu\,(R)}_{1loop}(q_1,q_2) - \Gamma^{(R)}_{1,1loop}(q^2=0)\,\gamma^\mu =$$

$$= \left\{ 1 + \left[F^{(R)}_{1,1loop}(q^2) - F^{(R)}_{1,1loop}(q^2=0) \right] \gamma^\mu + \frac{i\sigma^{\mu\nu} q_\nu}{2m_R} F^{(R)}_{2,1loop}(q^2) \right\} =$$

already UV-finite (F_2 is finite)

the UV-divergence problem is contained into $\lim q \to 0$ so we expand the term in [...]

$$= \left\{ \left[\cancel{F^{(R)}_{1,1loop}(q^2=0)} + \left.\frac{dF^{(R)}_{1,1loop}}{dq^2}\right|_{q=0} q^2 + \text{finite terms} - \cancel{F^{(R)}_{1,1loop}(q^2=0)} \right] + \textit{finite terms} \right\}$$

UV-finite

$$= \text{UV-finite terms}$$

Final considerations

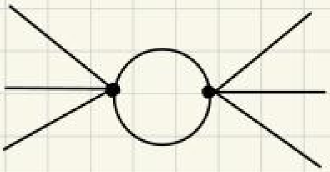
In a renormalizable theory we have the possibility to define a finite number of counter terms such as to cancel the finite number of UV divergences that arises order by order in perturbation theory. It's easy to see that this requirement fails if we consider a non-renormalizable theory.

Example: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda_5}{5!}\phi^5$

this theory has $D = 4 - \underbrace{N_\phi}_{\text{\# ext. scalar fields}} + \overset{\text{\# vertices}}{V}$

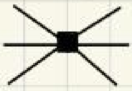
Suppose we want to apply the renormalization procedure we discussed for QED. The counter terms are in this case $\delta_1, \delta_m, \delta_{\lambda_5}$.

At one loop the diagram is



$N_\phi = 6$, $V = 2$ $\longrightarrow$ $D = 0$

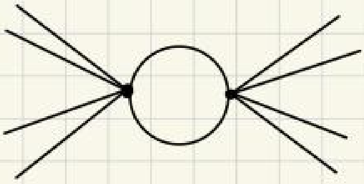
This diagram diverges!

In order to fix it I should have a counterterm  to fix it but we do not have it.

A possible way is to introduce an extra term in the original Lagrangian:

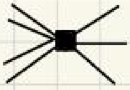
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda_5}{5!}\phi^5 - \frac{\lambda_6}{6!}\phi^6 \longrightarrow D = 4 - N_\phi + V_5 + 2V_6$$

We now ha a counterterm δ_6 that we can use to cancel the previous divergence however we now also have a one loop of the kind:



$N_\phi = 7$, $V_5 = 1$, $V_6 = 1$ $\rightarrow D = 0$

$\rightarrow$ diverges

In order to fix it I should have a counterterm  to fix it but we do not have it.

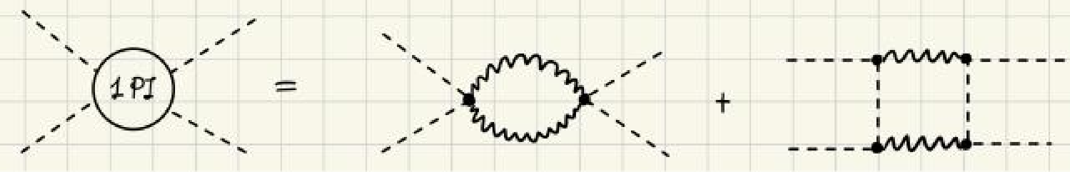
We could include an interaction $\frac{\lambda_7}{7!}\phi^7$ in the original Lagrangian, but we will need counter terms for interactions with higher values of ext N_ϕ, and the procedure never ends. We would need an infinite number of counterterms.

Example : Scalar QED theory (renormalizable) :

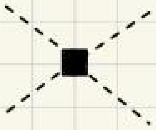
$$\mathcal{L}_{SQED} = \underbrace{(D_\mu \phi)^\dagger (D^\mu \phi)}_{} - m^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{this theory has} \quad D = 4 - N_\phi - N_\gamma$$

$$\hookrightarrow \begin{cases} e^2 A_\mu A^\mu \phi^\dagger \phi \\ ie A_\mu [(\partial_\mu \phi^\dagger)\phi - \phi^\dagger (\partial^\mu \phi)] \end{cases}$$

- Take the case in which $N_\phi = 4$, $N_\gamma = 0$ $\longrightarrow D = 0$



To cancel these divergences we need a counterterm of the type which is however not provided by our starting Lagrangian.

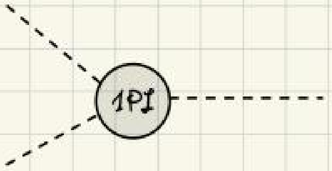


We should therefore consider to add another term:

$$\mathcal{L} = (D_\mu\phi)^+(D^\mu\phi) - m^2\phi^+\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4!}(\phi^+\phi)^2$$

that interaction term will provide the counterterm that cancels the divergences (it is totally consistent with the symmetries of the theory)

- Take the case in which $N_\gamma = 0$, $N_\phi = 3$



$\longrightarrow D = 1$ It would be divergent

However it is easy to see that such amplitude does not receive any contribution In other words it is not possible to write any diagram with 3 external scalar legs. To cancel its divergence, indeed, we would need a counterterm with 3 scalar fields and such a term is forbidden in our Lagrangian since it would break the U(1) symmetry : $(\phi^+\phi)\phi$ not U(1) symmetric.

The bottom line is that in a renormalizable theory we should write in our Lagrangian all renormalizable terms which are compatible with the underlying symmetries These theories are called UV-complete.

CONSEQUENCES OF RENORMALIZATION

Lezione 33 (pag 21-34)

SPONTANEOUS SYMMETRY BREAKING

We start from a classical discussion

SPONTANEOUS BREAKING OF AN SO(2) GLOBAL SYMMETRY

Consider the following scalar field theory

$$\boxed{\mathcal{L} = \sum_{K=1}^{2} \frac{1}{2}(\partial_\mu \phi_K)(\partial^\mu \phi_K) - \frac{m^2}{2}\sum_{K=1}^{2}\phi_K^2 - \frac{\lambda}{4}\left(\sum_{K=1}^{2}\phi_K^2\right)^2}$$

$\phi_1(x)$ and $\phi_2(x)$ are 2 real fields. The Lagrangian is invariant under a global SO(2) symmetry under which the scalar fields transform according to:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \longrightarrow \begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} = R(\theta)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \quad ; \quad R(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

If we introduce the vector $\vec{\phi} \equiv \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ the Lagrangian takes the form:

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})\cdot(\partial^\mu \vec{\phi}) - \frac{m^2}{2}\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2}$$

We now look for the field configuration of lowest energy. The discussion is classical but very often we use a "quantum language", and we say that we look for the vacuum state of the theory

The Hamiltonian of the theory is given by:

$$H = \int d^3\vec{x}\left[\sum_{K=1}^{2}\frac{1}{2}(\partial_t\phi_K)(\partial_t\phi_K) + \frac{1}{2}\sum_{K=1}^{2}\vec{\nabla}\phi_K\cdot\vec{\nabla}\phi_K + V(\phi_1,\phi_2)\right]$$

where:

$$V(\phi_1,\phi_2) = \frac{m^2}{2}(\phi_1^2+\phi_2^2) + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2$$

The first 2 terms are minimized to zero if we consider field configurations which are constant in time and homogeneus in space.

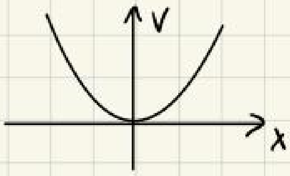
We need to minimize the potential energy: (we set $x \equiv |\vec{\phi}| = (\phi_1^2+\phi_2^2)^{1/2}$)

$$\longrightarrow \boxed{V(x) = \frac{m^2}{2}x^2 + \frac{\lambda}{4}x^4}$$

$$\begin{cases} V'(x) = m^2 x + \lambda x^3 = x(m^2+\lambda x^2) \overset{!}{=} 0 \\ V''(x) = m^2 + 3\lambda x^2 \end{cases}$$

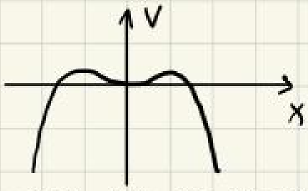
There are different cases:

i) $\boxed{m^2>0\ ,\ \lambda>0}$ ✓



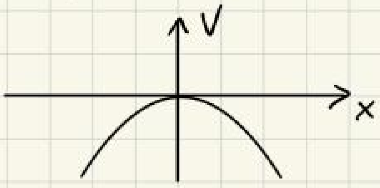
The potential has a minimum in $x=0$ (it is unique)

ii) $\boxed{m^2>0\ ,\ \lambda<0}$



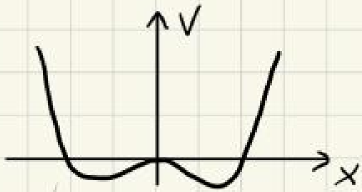
The potential has a minimum in $x=0$. However this is only local. The potential has a maximum at $x^2=-\frac{m^2}{\lambda}$ and the it is unbounded from below at large field values. We discard such situation.

iii) $\boxed{m^2<0\ ;\ \lambda<0}$



From the same reason we do not consider this case.

iv) $\boxed{m^2=-\mu^2\ ,\ \mu^2>0\ ;\ \lambda>0}$ ✓



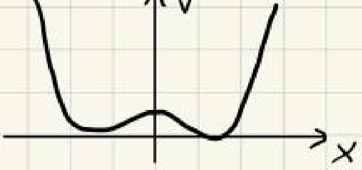
The mass has <u>apparently the wrong sign</u>

- If $x=0$ we now have $V''(x=0)=-\mu^2<0$ so that the potential now has a maximum in $x=0$.
- If $x^2=-\frac{m^2}{\lambda}=\frac{+\mu^2}{\lambda}$ we find $V''(x^2=\frac{\mu^2}{\lambda})=+2\mu^2>0$ so that the potential has a minimum in $\boxed{x^2=\mu^2/\lambda}$ and it is an absolute minimum
- The value of the potential at the minimum is given by :

$$V\left(x^2=\frac{\mu^2}{\lambda}\right)=-\frac{\mu^2}{2}\left(\frac{\mu^2}{\lambda}\right)+\frac{\lambda}{4}\left(\frac{\mu^4}{\lambda^2}\right)=\frac{-\mu^4}{2\lambda}+\frac{\mu^4}{4\lambda}=-\frac{\mu^4}{4\lambda}\neq 0$$

- In order to have a vacuum state with zero energy we conventionally add to the initial potential the constant term $+\frac{\mu^4}{4\lambda}$. Our potential now reads :

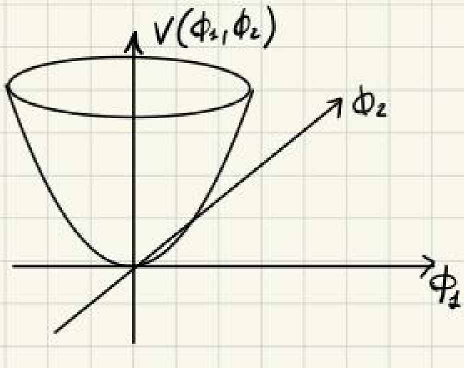
$$\boxed{V(x)=-\frac{\mu^2}{2}x^2+\frac{\lambda}{4}x^4+\frac{\mu^4}{4\lambda}}$$



<u>Consider now the 2 cases in the field space</u>

- $\boxed{m^2>0\ ,\ \lambda>0}$ $\longrightarrow$ minimum at $\phi_1=\phi_2=0$

The "vacuum" is unique and, most importantly, it's invariant under the $SO(2)$ global symmetry. This is a trivial statement since the vacuum is $\begin{pmatrix}0\\0\end{pmatrix}$ if we apply any $SO(2)$ rotation we still get $\begin{pmatrix}0\\0\end{pmatrix}$

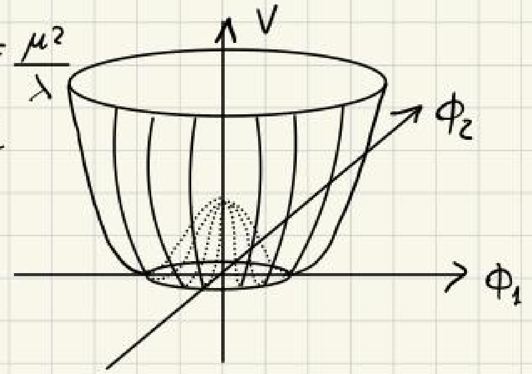


This case therefore corresponds to the situation in which the symmetry is realized à la Wigner Weyl Precisely because the vacuum is unique and it's left invariant under the action of the symmetry group.
In this case the Lagrangian is manifestly invariant under the action of the symmetry group and furthermore the symmetry is also manifest in the particle spectrum of the theory.
In fact, if we now interpret

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\vec{\phi})\cdot(\partial^\mu\vec{\phi}) - \frac{m^2}{2}\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2$$

as a Q.F.T., it describes 2 scalar particles (with precisely the same mass)

- $\boxed{m^2 \equiv -\mu^2, \mu^2>0 \ ; \lambda>0} \longrightarrow$ minimum at $\phi_1^2+\phi_2^2 = \frac{\mu^2}{\lambda}$

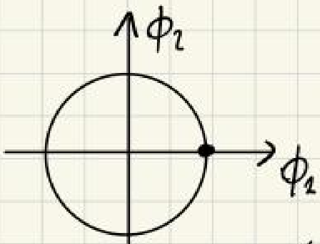


The vaccum is not unique, indeed the relation $\phi_1^2+\phi_2^2 = \frac{\mu^2}{\lambda}$ identifies a manifold of vacua : the minimum accurs for any ϕ_1, ϕ_2 that lie on a circle of radius $\sqrt{\mu^2/\lambda}$

We have therefore a degeneracy i.e. a continuum of ground states, all with the same energy and all connected via SO(2) rotations

The system chooses one of the degenerate vacua. Suppose you keep a ball at the maximum and let it go. The ball will fall towards the bottom of the potential and will eventually come to rest at a specific point on the circle. Which point we choose is arbitrary, however once a choic is made, the ground state loses the rotational invariance. We say that we have a spontaneous breakdown of the symmetry.

Let's make a choice, we consider for instance the point $\phi_1 = \sqrt{\mu^2/\lambda}$, $\phi_2 = 0$. We introduce the following notation:



$$v^2 \equiv \frac{\mu^2}{\lambda} \quad ; \quad \langle\vec{\phi}\rangle \equiv \begin{pmatrix} v \\ 0 \end{pmatrix} \qquad \langle\phi_1\rangle = v \ ; \ \langle\phi_2\rangle = 0$$

We say that the field has a vaccum expectation value $\langle\vec{\phi}\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix}$

This specific choice of the vacuum <u>is not invariant</u> under rotations since now $R(\theta)\begin{pmatrix} v \\ 0 \end{pmatrix} \neq \begin{pmatrix} v \\ 0 \end{pmatrix}$

Although our discussion is classical, it seems we are in a situation conceptually different compared to the case in which the symmetry is realized à la Wigner - Weyl. In fact, in this case the vacuum is not invariant under the action of the symmetry.

CONSEQUENCES OF SPONTANEOUS SYMMETRY BREAKING

We define the shifted fields by writing :

$$\boxed{\begin{aligned} \phi_1(x) &\equiv v + \chi(x) \\ \phi_2(x) &\equiv \theta(x) \end{aligned}}$$

such that $\langle\chi(x)\rangle = 0$ (using the same notation as before). Therefore the Lagrangian can be rewritten in terms of χ and θ.

KINETIC TERM

Since v is a constant it does not affect these derivative terms:

$$\mathcal{L}_{kin} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) = \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) + \frac{1}{2}\partial_\mu \theta \, \partial^\mu \theta$$

POTENTIAL TERM

$$V(\phi_1, \phi_2) = \frac{-\mu^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 + \frac{\mu^4}{4\lambda} =$$

$$= -\frac{\mu^2}{2}\left[(v+\chi)^2 + \theta^2\right] + \frac{\lambda}{4}\left[(v+\chi)^2 + \theta^2\right]^2 + \frac{\mu^4}{4\lambda} =$$

$$= -\frac{\mu^2}{2}\left[v^2 + \chi^2 + 2v\chi + \theta^2\right] + \frac{\lambda}{4}\left[v^4 + \chi^4 + 4v^2\chi^2 + \theta^4 + 2v^2\chi + 4v^3\chi + 2v^2\theta^2 + 4v\chi^3 + 2\chi^2\theta^2 + 4v\chi\theta^2\right] + \frac{\mu^4}{4\lambda} =$$

Zero-th order terms in the fields

$$\frac{\mu^2 v^2}{2} + \frac{\lambda v^4}{4} + \frac{\mu^4}{4\lambda} = \frac{-\mu^2}{2}\left(\frac{\mu^2}{\lambda}\right) + \frac{\lambda}{4}\left(\frac{\mu^4}{\lambda^2}\right) + \frac{\mu^4}{4\lambda} = -\frac{\mu^4}{2\lambda} + \frac{\mu^4}{4\lambda} + \frac{\mu^4}{4\lambda} = 0$$

Terms linear in the fields

$$-\frac{\mu^2}{2}(2v\chi) + \frac{\lambda}{4}(4v^3\chi) = -\mu^2 v\chi + \lambda v^3 \chi = v\underbrace{(-\mu^2 - \lambda v^2)}_{=0}\chi = 0$$

Quadratic terms in the fields

$$-\frac{\mu^2}{2}\theta^2 - \frac{\mu^2}{2}\chi^2 + \frac{\lambda}{4}\left(4v^2\chi^2 + 2v^2\chi^2 + 2v^2\theta^2\right) =$$

$$= \chi^2\left(-\frac{\mu^2}{2} + \frac{\lambda}{4}6v^2\right) - \frac{\mu^2}{2}\theta^2 + \frac{\lambda}{4}2v^2\theta^2 =$$

$$= \frac{\chi^2}{2}\left(-\mu^2 + \frac{3\lambda\mu^2}{\lambda}\right) + \frac{\theta^2}{2}\left(-\mu^2 + \frac{\lambda\mu^2}{\lambda}\right) = \frac{\chi^2}{2}(2\mu^2)$$

Cubic and quartic terms

$$\frac{\lambda}{4}\left(\chi^4 + \theta^4 + 4v\chi^3 + 2\chi^2\theta^2 + 4v\chi\theta^2\right)$$

We arrive at the Lagrangian in the broken phase:

$$\boxed{\begin{aligned} \mathcal{L} = {} & \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \frac{1}{2}(2\mu^2)\chi^2 + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) \\ & - \frac{\lambda}{4}\left(\chi^4 + \theta^4 + 4v\chi^3 + 2\chi^2\theta^2 + 4v\chi\theta^2\right) \end{aligned}}$$

Free Lagrangian

Interaction $\mathcal{L}$.

If we look at the quadratic Lagrangian piece we see an important fact:

- The field $\chi(x)$ has mass $m_\chi^2 \equiv 2\mu^2$
- The field $\theta(x)$ has mass $m_\theta^2 = 0$

We see a crucial difference compared to the case in which the symmetry is realized à la Wigner Weyl. In the presence of spontaneous symmetry breaking the symmetri (in this case

SO(2)) is not realized in the mass spectrum of the theory since we now have, instead of a doublet of scalar fields with the same mass, one massive field and one massless field.
In the Wigner-Weyl case, the symmetry is manifest at the level of the Lagrangian.
What happens in the case of spontaneous symmetry breaking? If we look at the Lagrangian we wrote before there seems to be no sign of SO(2) symmetry

Let's consider the same Lagrangian but with a different choice of field variables $\varphi(x)$:

$$\mathcal{L} = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - V(\varphi) \qquad V(\varphi) = -\mu^2 \varphi^* \varphi + \lambda (\varphi^* \varphi)^2 + \frac{\mu^4}{4\lambda}$$

which is equivalent to our original Lagrangian if we set:

$$\boxed{\varphi(x) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + i \phi_2(x) \right)}$$

The Lagrangian is symmetric under phase redefinitions

$$\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) e^{-i\alpha} \qquad SO(2) \cong U(1)$$

(We're exploiting the isomorphism between SO(2) and U(1))

This symmetry is spontaneously broken since we have degenerate vacua described by the condition:

$$\varphi^* \varphi = \frac{\mu^2}{2\lambda}$$

In this variables the physics of spontaneous symmetry breaking is more transparent.

Let us write:

$$\varphi(x) = \underbrace{\frac{1}{\sqrt{2}} \left[\phi_1(x) + i \phi_2(x) \right]}_{\substack{\text{cartesian representation} \\ \text{field space}}} = \underbrace{\frac{1}{\sqrt{2}} \rho_1(x) e^{i \rho_2(x)}}_{\substack{\text{exponential representation;} \\ \text{both } \rho_1 \text{ and } \rho_2 \text{ are real fields}}}$$

We choose the vacuum. Our previous choice was $\langle \phi_1 \rangle = v$, $\langle \phi_2 \rangle = 0$. In the new variables we set $\langle \rho_1 \rangle = v$, $\langle \rho_2 \rangle = 0$. We introduce the shifted fields:

$$\boxed{\begin{aligned} \rho_1(x) &\equiv h(x) + v \\ \rho_2(x) &\equiv \frac{\pi(x)}{f_\pi} \end{aligned}} \longrightarrow \boxed{\varphi(x) = \frac{1}{\sqrt{2}} \left[h(x) + v \right] \exp\left(\frac{i \pi(x)}{f_\pi} \right)}$$

Where f_π is a constant with dimension [M] in natural units to make [π] of dimension 1. $h(x)$ and $\pi(x)$ are real scalar fields of mass-dimension 1.

We now go back to the Lagrangian: let's rewrite it in terms of h and π

<u>POTENTIAL TERM</u>

$$\begin{aligned} V(\varphi) &= -\frac{\mu^2}{2} \left(h^2 + v^2 + 2hv \right) + \frac{\lambda}{4} \left(h^2 + v^2 + 2hv \right)^2 + \frac{\mu^4}{4\lambda} = \\ &= -\frac{\mu^2}{2} \left(h^2 + v^2 + 2hv \right) + \frac{\lambda}{4} \left(h^4 + v^4 + 4h^2v^2 + 2h^2v^2 + 4hv^3 + 4vh^3 \right) + \frac{\mu^4}{4\lambda} \end{aligned}$$

Zero-th order terms in the fields

$$-\frac{\mu^2 v^2}{2}+\frac{\lambda}{4}v^4+\frac{\mu^4}{4\lambda} = -\frac{\mu^2}{2}\left(\frac{\mu^2}{\lambda}\right)+\frac{\lambda}{4}\left(\frac{\mu^4}{\lambda}\right)+\frac{\mu^4}{4\lambda}=0$$

Terms linear in the fields

$$-\frac{\mu^2}{2}(2hv)+\frac{\lambda}{4}(4hv^3) = hv(-\mu^2+\lambda v^2)=0$$

Terms quadratic in the fields

$$-\frac{\mu^2}{2}h^2+\frac{\lambda}{4}(6h^2v^2) = \frac{1}{2}h^2(-\mu^2+3\lambda v^2)=\frac{1}{2}h^2(-\mu^2+3\mu^2)=\frac{1}{2}(2\mu^2)h^2$$

Cubic and quartic terms

$$\frac{\lambda}{4}(h^4+4vh^3) = \frac{\lambda}{4}h^4+(\lambda v)h^3$$

$$\longrightarrow \boxed{V(h,\pi)=\frac{1}{2}(2\mu^2)h^2+\frac{\lambda}{4}h^4+(\lambda v)h^3}$$

N.B. The field π is guaranteed to be massless since it never appears in the potential.

KINETIC TERMS

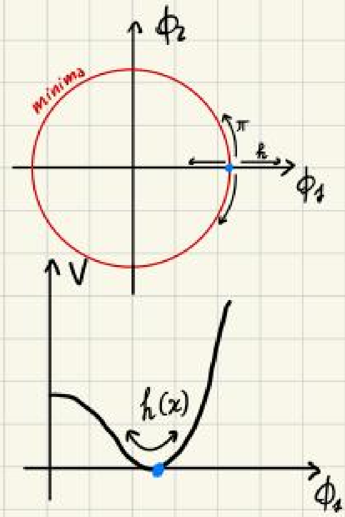
$$\begin{aligned}(\partial_\mu\varphi^*)(\partial^\mu\varphi) &= \frac{1}{2}\partial_\mu\left((h+v)e^{-\frac{i\pi}{f_\pi}}\right)\partial^\mu\left((h+v)e^{\frac{i\pi}{f_\pi}}\right)=\\
&=\frac{1}{2}\left[(\partial_\mu h)e^{-\frac{i\pi}{f_\pi}}+(h+v)\frac{(-i)}{f_\pi}(\partial_\mu\pi)e^{-\frac{i\pi}{f_\pi}}\right]\left[(\partial^\mu h)e^{\frac{i\pi}{f_\pi}}+(h+v)\frac{i}{f_\pi}(\partial^\mu\pi)e^{\frac{i\pi}{f_\pi}}\right]=\\
&=\frac{1}{2}(\partial_\mu h)(\partial^\mu h)+\cancel{\frac{i}{2f_\pi}(\partial_\mu h)(\partial^\mu\pi)(h+v)}+\cancel{\frac{(-i)}{2f_\pi}(h+v)(\partial_\mu\pi)(\partial^\mu h)}+\frac{1}{2}\frac{(h+v)^2}{f_\pi^2}(\partial_\mu\pi)(\partial^\mu\pi)\\
&=\underbrace{\frac{1}{2}(\partial_\mu h)(\partial^\mu h)+\frac{v^2}{2f_\pi^2}(\partial_\mu\pi)(\partial^\mu\pi)}_{\text{Pure Kinetic terms}}+\underbrace{\frac{1}{2f_\pi^2}(h^2+2hv)(\partial_\mu\pi)(\partial^\mu\pi)}_{\text{Interactio between } h \text{ and } \pi}\end{aligned}$$

If we set $f_\pi = v$, the Kinetic term for $\pi(x)$ is canonically normalized.
All in all we find :

$$\boxed{\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu h)(\partial^\mu h)-\frac{1}{2}(2\mu^2)h^2+\frac{1}{2}(\partial_\mu\pi)(\partial^\mu\pi)\\
+\frac{1}{2v^2}(h^2+2hv)(\partial_\mu\pi)(\partial^\mu\pi)-\lambda v h^3-\frac{\lambda}{4}h^4\end{aligned}}$$

As before we find a massive scalar field $h(x)$ with mass $m_h^2 \equiv 2\mu^2$ and a massless scalar field $\pi(x)$. The fact that $\pi(x)$ is massless has two interpretations.

i) Intuitive graphical interpretation



- $\pi(x)$ describes angular fluctuations. Along this direction there is no potential.

- $h(x)$ describes radial fluctuations. Along this direction we have a potential. We call often h the "radial mode".

ii) Interpretation in terms of a non linearly realized symmetry

Consider the Lagrangian written in terms of $\pi(x)$ and $h(x)$. This Lagrangian has a symmetry. Consider the transformation:

$$\boxed{\begin{aligned} h(x) &\longrightarrow h(x) \\ \pi(x) &\longrightarrow \pi(x) + const \end{aligned}}$$

This is a symmetry of $\mathcal{L}$ since:

i) <u>$h(x)$ does not transform</u>

ii) <u>$\pi(x)$ enters in $\mathcal{L}$ only derivatively</u> : consequently the shift $\pi \rightarrow \pi + const$ leaves the Lagrangian invariant. The fact that $\pi(x)$ has no mass term of the form $\pi(x)^2$ and, in general, any potential term of the kind $\pi(x)^3, \pi(x)^4$ is inextricably link to this shift symmetry.

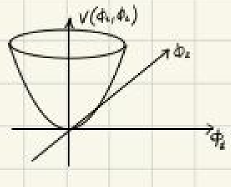
<u>What is the origin of this symmetry?</u>

Consider our definition : $\varphi(x) = \frac{1}{\sqrt{2}}\left[h(x)+v\right]\exp\left(\frac{i\,\pi(x)}{v}\right)$ and take the transformation. In terms of $\varphi(x)$ we get:

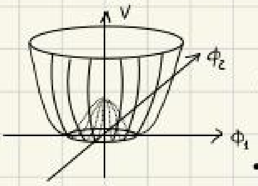
$$\varphi(x) \longrightarrow \frac{1}{\sqrt{2}}\left[h(x)+v\right] e^{\frac{i\pi}{v}} e^{\frac{ic}{v}} \equiv \varphi(x)\, e^{-i\alpha} \qquad \text{where } \alpha \equiv -\frac{c}{v}$$

This is nothing but our original $SO(2) \cong U(1)$ symmetry, but now realized as a shift symmetry on the massless field $\pi(x)$. This is an important point : the symmetry $SO(2) \cong U(1)$ is still present in the Lagrangian even if we consider the case of spontaneous symmetry breaking, it is only realized in a different way.

We call the field $\pi(x)$: Goldstone boson. It remains massless in the presence of spontaneous symmetry breaking.
Let's make a short summary of what we've learned so far.



- Wigner Weyl realization
- Symmetry realized linearly on the dynamical fields of the theory $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \rightarrow R(\theta)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$
- Unique vacuum state, invariant under the symmetry.
- Symmetry realized in the particle spectrum.



- Sp. symmetry breaking
- Degeneracy of vacua, the ground state is not invariant under the symm.
- The symmetry is not realized in the mass spectrum
- Emergence of massless field, the Goldstone boson.
- Symm. realized non linearly in the $\mathcal{L}$.

- We can use this model to discuss another peculiar property of Goldstone bosons. Consider first the Lagrangian

$$\boxed{\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \frac{1}{2}(2\mu^2)\chi^2 + \frac{1}{2}(\partial_\mu\theta)(\partial^\mu\theta) \\ -\frac{\lambda}{4}\left(\chi^4+\theta^4+4v\chi^3+2\chi^2\theta^2+4v\chi\theta^2\right)\end{aligned}}$$

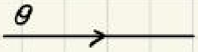
Compute the scattering process (at the tree level):

$$\boxed{\chi(P_1)+\theta(P_2) \longrightarrow \chi(P_3)+\theta(P_4)}$$

We use the following Feynman rules:

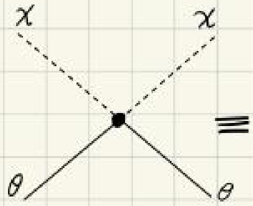
- **PROPAGATORS**:

$= \dfrac{i}{p^2-m_\chi^2+i\varepsilon}$;

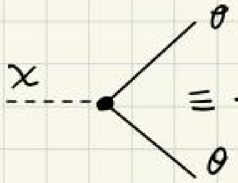


$= \dfrac{i}{p^2+i\varepsilon}$

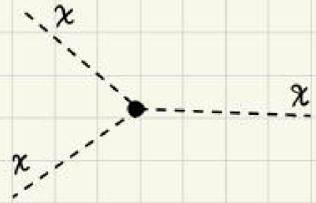
- **VERTEX**

- 

$$\equiv i\left(-\frac{\lambda}{4}\right)\cdot 2\,(2)(2) = -2i\lambda$$

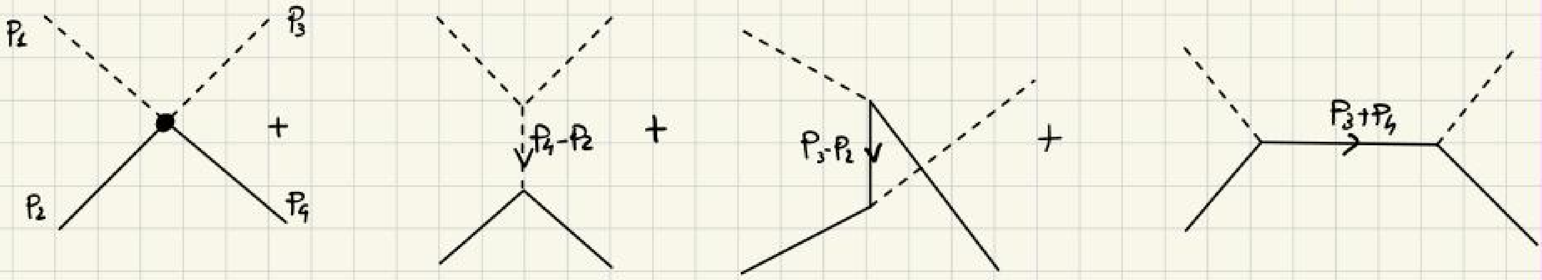
- 

$$\equiv -2i\lambda v$$

- 

$$\equiv i\left(\frac{-\lambda}{4}\right)4v\,3! = -6i\lambda$$

And now let's compute the amplitude. We'll have the contribution by 4 diagrams:



$$i\mathcal{M} = (-2i\lambda) + (-6i\lambda v)\frac{i}{(P_4-P_2)^2-m_\chi^2+i\varepsilon}(-2i\lambda v) +$$

$$+(-2i\lambda v)\frac{i}{(P_3-P_2)^2+i\varepsilon}(-2i\lambda v) + (-2i\lambda v)\frac{i}{(P_3+P_4)^2+i\varepsilon}(-2i\lambda v) =$$

$$= -2i\lambda + \frac{(-i)\,12\lambda^2v^2}{(P_4-P_2)^2-m_\chi^2} + \frac{(-i)\,4\lambda^2v^2}{(P_3-P_2)^2} + \frac{(-i)\,4\lambda^2v^2}{(P_3+P_4)^2}$$

i) Let's take the χ on the mass-shell (physical particles) $p_1^2 = p_3^2 = m_\chi^2$

ii) Let's consider $p_2^\mu, p_4^\mu \xrightarrow{\lim} 0$ (4-mom of the Goldstone boson $\to 0$)

$$\longrightarrow i\mathcal{M} = (-2i\lambda) + \frac{(-i)\,12\lambda^2 v^2}{-m_\chi^2} + \frac{(-i)4\lambda^2 v^2}{m_\chi^2} + \frac{(-i)4\lambda^2 v^2}{m_\chi^2}$$

$$= (-2i\lambda) + \frac{i4\lambda^2 v^2}{m_\chi^2} = -2i\lambda + \frac{i4\lambda^2 v^2}{2\mu^2\lambda} = 0$$

So we find that the scattering amplitude vanishes in the limit in which the momentum of the Goldstone bosons goes to zero. This property however is not manifest at the level of the Lagrangian since it comes from a non-trivial cancellation among diagrams.

- Consider now the theory described by the Lagrangian

$$\boxed{\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}(2\mu^2)h^2 + \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) \\ & + \frac{1}{2v^2}(h^2 + 2hv)(\partial_\mu \pi)(\partial^\mu \pi) - \lambda v h^3 - \frac{\lambda}{4}h^4\end{aligned}}$$

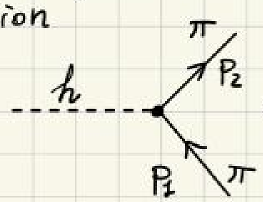
and consider again the scattering among the massive field - now $h(x)$ - and the massless Goldstone - now $\pi(x)$

$$\boxed{h(p_1) + \pi(p_2) \longrightarrow h(p_3) + \pi(p_4)}$$

In this case even without do the computation it is evident that this amplitude vanishes if we take the limit in which $p_2, p_4 \longrightarrow 0$

Because of the non-linearly realized symmetry the Goldstone field is only coupled derivatively; for instance, consider the cubic interaction

$$\mathcal{L}_{int} = \frac{h}{v}(\partial_\mu \pi)(\partial^\mu \pi) \longrightarrow$$



$$\equiv \frac{i}{v}(2p_1 \cdot p_2)$$

and all these interactions vanish as the Goldstone momentum goes to zero.

Consider a short summary so far. We can describe the physics of spontaneous symmetry breaking with the two lagrangians

$$\boxed{\begin{aligned}\mathcal{L}_\theta = {} & \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \frac{1}{2}(2\mu^2)\chi^2 + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) \\ & - \frac{\lambda}{4}(\chi^4 + \theta^4 + 4v\chi^3 + 2\chi^2\theta^2 + 4v\chi\theta^2)\end{aligned}}$$

$$\boxed{\begin{aligned}\mathcal{L}_\pi = {} & \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}(2\mu^2)h^2 + \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) \\ & + \frac{1}{2v^2}(h^2 + 2hv)(\partial_\mu \pi)(\partial^\mu \pi) - \lambda v h^3 - \frac{\lambda}{4}h^4\end{aligned}}$$

→ This description, the non-linear one, makes the properties of the Goldstone boson more manifest. In particular they are linked to the non linear realization of the symmetry broken by the vacuum

The 2 formulations are related by a change of variables in field space.

$$\frac{1}{\cancel{\sqrt{2}}}\left[v+\chi+i\theta\right] \overset{!}{=} \frac{1}{\cancel{\sqrt{2}}}\left[h+v\right]\exp\left(i\frac{\pi}{v}\right)$$
$$= (h+v)\left(1+\frac{\pi}{v}-\frac{\pi^2}{2v^2}-\frac{i\pi^3}{6v^3}+---\right)$$
$$= \left(h+v-\frac{\pi^2}{2v}+\ldots\right)+i\left(\pi+\frac{\pi h}{v}+\ldots\right)$$

There fore :

$$\boxed{\begin{aligned}\chi(x) &= h(x)-\frac{\pi(x)^2}{2v}+\ldots\\ \theta(x) &= \pi(x)+\frac{\pi(x)h(x)}{v}+\ldots\end{aligned}}$$

Non-linear field redefinition.

At the linear order, we just have $\chi(x)\equiv h(x)$ and $\theta(x)\equiv\pi(x)$. This is because the two theories have the same *free-theory limit*. (Kinetic terms)

The two theories give the same results if we consider the computation of on-shell scattering matrix elements.

Exercise

Let's take the scattering for this process:

$$\chi(P_1)+\theta(P_2)\longrightarrow\chi(P_3)+\theta(P_4)$$

Put the ext. particles on shell. Compute the amplitude and compare the result with the other basis. $(h\pi\longrightarrow h\pi)$.

$$i\mathcal{M} = -2i\lambda+\frac{(i)12\lambda^2v^2}{(P_1-P_2)^2-m_\chi^2}+\frac{(-i)4\lambda^2v^2}{(P_3-P_1)^2}+\frac{(-i)4\lambda^2v^2}{(P_3+P_4)^2}$$
$$= -2i\lambda+\frac{(-i)12\lambda^2v^2}{-2P_2\cdot P_4-m_\chi^2}+\frac{(-i)4\lambda^2v^2}{-2P_2\cdot P_3+m_\chi^2}+\frac{(-i)4\lambda^2v^2}{2P_3\cdot P_4+m_\chi^2}$$

$P_1^2=P_3^2=m_\chi^2=2\mu^2$

$P_2^2=P_4^2=0$

Defining the Mandelstan variables:

$$\boxed{\begin{aligned}s &= (P_1+P_2)^2=(P_3+P_4)^2=m_\chi^2+2P_3\cdot P_4\\ t &= (P_1-P_3)^2=(P_2-P_4)^2=-2P_2\cdot P_4\\ u &= (P_1-P_4)^2=(P_2-P_3)^2=m_\chi^2-2P_2\cdot P_3\end{aligned}}$$

$$s+t+u=2m_\chi^2$$

we can rewrite $i\mathcal{M}$ as:

$$i\mathcal{M} = -2i\lambda+\frac{(-i)12\lambda^2v^2}{t-m_\chi^2}+\frac{(-i)4\lambda^2v^2}{2m_\chi^2-s-t}+\frac{(-i)4\lambda^2v^2}{s}$$

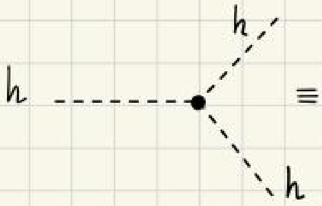
Finally, it is useful to trade λv^2 for m_χ^2 using $m_\chi^2=2\mu^2=2\lambda v^2$:

$$\boxed{i\mathcal{M} = -2i\lambda+\frac{(-i)6\lambda m_\chi^2}{t-m_\chi^2}+\frac{(-i)2\lambda m_\chi^2}{u}+\frac{(-i)2\lambda m_\chi^2}{s}}$$

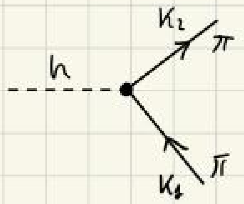
We now consider the same computation with the non-linear realization

$$h(P_1) + \pi(P_2) \longrightarrow h(P_3) + \pi(P_4)$$

The rules we need are:

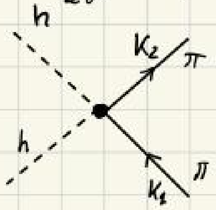


$$\equiv -i\lambda v 6$$



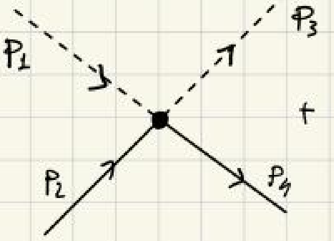
$$\equiv \frac{i}{v}(2K_1\cdot K_2)$$

and in addition, from $\mathcal{L}_{int} = \frac{1}{2v^2}h^2(\partial_\mu\pi)(\partial^\mu\pi)$ we also have:



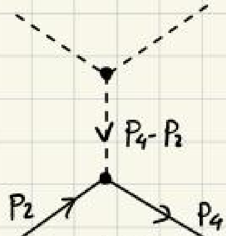
$$= \frac{i}{2v^2}(2)(-iK_1^\mu)(iK_2^\mu)2 = \frac{i}{v^2}(2K_1\cdot K_2)$$

The amplitude is described by the following diagrams:



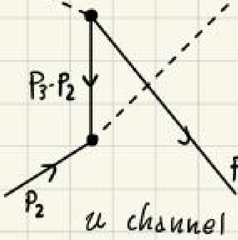
contact interaction

+



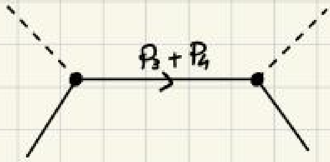
t channel

+



u channel

+



s-channel

$$i\mathcal{M} = \frac{i}{v^2}(2P_2\cdot P_4) + (-i6\lambda v)\frac{i}{t-m_\chi^2}\frac{i}{v}(2P_2\cdot P_4) + \frac{i}{v}2\underbrace{(-P_3+P_2)}_{P_4-P_1}\cdot P_4\cdot\frac{i}{u}\frac{i}{v}2(P_2\cdot(P_2-P_3)) +$$

$$+\frac{i}{v}2P_2\cdot\underbrace{(P_3+P_4)}_{P_1+P_2}\frac{i}{s}\frac{i}{v}2(P_3+P_4)\cdot P_4 =$$

$$= \frac{i}{v^2}(-t) + \frac{i6\lambda(-t)}{t-m_\chi^2} + \frac{(-i)}{v^2}\frac{(-2P_1\cdot P_4)(-2P_2\cdot P_3)}{u} + \frac{(-i)}{v^2}\frac{(2P_1\cdot P_2)(2P_3\cdot P_4)}{s} =$$

$$= \frac{i}{v^2}(-t) - \frac{i6\lambda t}{t-m_\chi^2} - \frac{i}{v^2u}(u-m_\chi^2)(u-m_\chi^2) - \frac{i}{sv^2}(s-m_\chi^2)(s-m_\chi^2)$$

We note that in this amplitude there are terms that, if taken individually, are very dangerous since they grow up with energy. However if we sum we get (considering only the terms which grow with energy):

$$-\frac{it}{v^2} - \frac{iu}{v^2} - \frac{is}{v^2} = \frac{-i}{v^2}(s+t+u) = -\frac{i}{v^2}(2m_\chi^2)$$

and the amplitude reads:

$$i\mathcal{M} = -\frac{2im_\chi^2}{v^2} - \frac{6i\lambda t}{t-m_\chi^2} - \frac{i}{v^2u}(m_\chi^4 - 2um_\chi^2) - \frac{i}{sv^2}(m_\chi^4 - 2sm_\chi^2) =$$

$$= -\frac{2im_\chi^2}{v^2} - \frac{im_\chi^4}{uv^2} - \frac{im_\chi^4}{sv^2} - \frac{6i\lambda t}{t-m_\chi^2} + \frac{2ium_\chi^2}{uv^2} + \frac{i}{sv^2}2sm_\chi^2 =$$

$$= -\frac{4i\lambda v^2}{v^2} - \frac{i m_x^2 2\lambda v^2}{u v^2} - \frac{i m_x^2 2\lambda v^2}{s v^2} - \frac{6i\lambda (t - m_x^2 + m_x^2)}{t - m_x^2} + \frac{4 i m_x^2}{v^2} =$$

$$= -4i\lambda - \frac{i m_x^2 2\lambda}{u} - \frac{i m_x^2 2\lambda}{s} - 6i\lambda + \frac{4i(2\lambda \cancel{v^2})}{\cancel{v^2}} - \frac{6i\lambda m_x^2}{t - m_x^2} =$$

$$= -4i\lambda - \frac{i m_x^2 2\lambda}{u} - \frac{i m_x^2 2\lambda}{s} + 2i\lambda - \frac{6i\lambda m_x^2}{t - m_x^2}$$

$$\longrightarrow \boxed{i\mathcal{M} = -2i\lambda + \frac{(-i)\cdot 6\lambda m_x^2}{t - m_x^2} + \frac{(-i) 2\lambda m_x^2}{u} + \frac{(-i) 2\lambda m_x^2}{s}}$$

Even though the 2 Lagrangians are completely different, the 2 theories give the same on-shell S-matrix elements (that is, they describe the same physics). The reason is that the 2 theories are related by a field redefinition that does not alter the free theory limit.
We remark that the equivalence between the 2 theories is only true on-shell. In particular there is a price to pay for making the Goldstone boson shift symmetry explicit. The Lagrangian $\mathcal{L}_\pi$ has non-renormalizable interactions; this is unavoidable if we start from a renormalizable theory ($\mathcal{L}_\theta$ in our case) and make a non linear field redefinition.

Which choice, then, is the best one? It depends on which kind of computation we are interested in.

- If we are interested in highlighting some structural properties of the theory, then we tipically choose a non-linear redefinition.
- If we want to perform complicated loop calculations, then we choose the cartesian basis.

FURTHER PROPERTIES : THE "BROKEN" CURRENT

Consider the current J^μ associated to the broken symmetry:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \longrightarrow \begin{pmatrix}\phi_1'\\ \phi_2'\end{pmatrix} = R(\theta)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}1 & -\theta\\ \theta & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + O(\theta^2) = \begin{pmatrix}\phi_1 - \theta\phi_2\\ \theta\phi_1 + \phi_2\end{pmatrix} \longrightarrow \boxed{\begin{matrix} D\phi_1 = -\phi_2 \\ D\phi_2 = +\phi_1 \end{matrix}}$$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} D\phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} D\phi_2 = (\partial^\mu \phi_1)(-1)\phi_2 + (\partial^\mu \phi_2)\phi_1 =$$

$$\phi_1 = v + \chi$$
$$\phi_2 = \theta$$

$$= (\partial^\mu \chi)(-\theta) + (\partial^\mu \theta)(v + \chi) = v(\partial^\mu \theta) + \chi(\partial^\mu \theta) - \theta(\partial^\mu \chi)$$

$$\longrightarrow \boxed{J^\mu = \underbrace{v(\partial^\mu \theta)}_{\text{the current has a term linear in the Goldstone field}} + \chi(\partial^\mu \theta) - \theta(\partial^\mu \chi)}$$

We consider the matrix element $\langle \Omega | J^\mu | \vec{p} \rangle$. We compute it in the free theory limit:

$$\langle 0 | \left[\underline{v(\partial^\mu \theta)} + \chi(\partial^\mu\theta) - \theta(\partial^\mu \chi)\right] \underline{|\vec{p}\rangle}$$

Only this term contributes.
We write the Goldstone field as:

$$\theta(x) = \int \frac{d^3\vec{x}}{(2\pi)^3\sqrt{2E_K}}\left[a(\vec{K})e^{-i\vec{K}\cdot\vec{x}} + a^\dagger(\vec{K})e^{i\vec{K}\cdot\vec{x}}\right]$$

One specific particle state of the Goldstone field:
$|\vec{p}\rangle = \sqrt{2E_p}\, a^\dagger(p)|0\rangle$; $E_p = |\vec{p}|$
since $m_\theta = 0$.

$$= v\langle 0 | \int \frac{d^3\vec{K}}{(2\pi)^3\sqrt{2E_K}}\left(a(\vec{K})(-iK^\mu)(e^{-iK\cdot x}) + a^\dagger(\vec{K})\, iK^\mu e^{iK\cdot x}\right)\cdot \sqrt{2E_p}\, a^\dagger(\vec{p})|0\rangle =$$

$$= v\int \frac{d^3\vec{K}}{(2\pi)^3\sqrt{2E_K}}(-iK^\mu)\, e^{-iK\cdot x}\sqrt{2E_p}\,(2\pi)^3\,\delta(\vec{x}-\vec{p}) = -ip^\mu e^{-ip\cdot x}v$$

$$\longrightarrow \boxed{\langle\vec{p}|J^\mu(x)|0\rangle = ip^\mu e^{ip\cdot x}v}$$

The "broken" current $J^\mu(x)$ creates a single particle Goldstone state from the vacuum.

The result in the full interacting theory can be guessed as follows. Consider $x=0$ $\langle\vec{p}|J^\mu(0)|0\rangle = ivp^\mu$. In the full theory $\langle\vec{p}|J^\mu(0)|\Omega\rangle$ is a four-vector and it must be prop. to the only four-vector we have at our disposal p^μ. We simply write:

$$\boxed{\langle\vec{p}|J^\mu(x)|\Omega\rangle = iFp^\mu} \qquad \text{where } F = \text{const.}$$

At non-zero x, we just use $e^{-ia_\mu P^\mu}J^\mu(x)e^{ia_\mu P^\mu} = J^\mu(x-a)$ that is $J^\mu(0) = e^{-ixP}J^\mu(x)e^{ixP}$

$$\longrightarrow \langle\vec{p}|J^\mu(0)|\Omega\rangle = \langle\vec{p}|e^{-ix\cdot P}J^\mu(x)e^{ix\cdot P}|\Omega\rangle = e^{-ix\cdot p}\langle\vec{p}|J^\mu(x)|\Omega\rangle$$

so that:

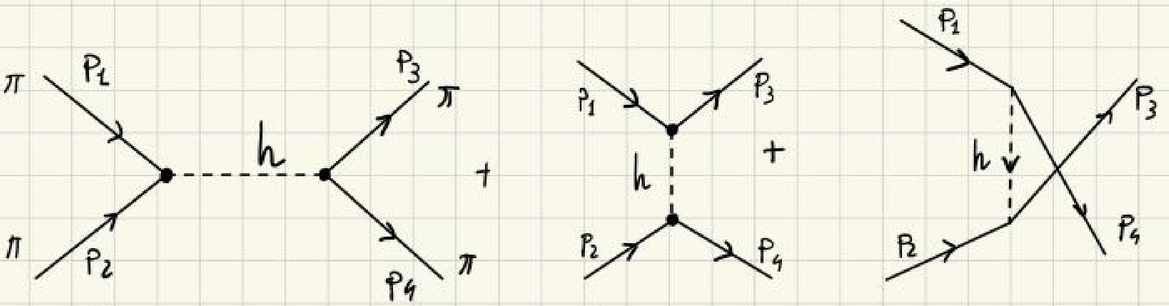
$$\boxed{\langle\vec{p}|J^\mu(x)|\Omega\rangle = iFp^\mu e^{ix\cdot p}}$$

FURTHER PROPERTIES: EFT FOR GOLDSTONE BOSONS

Consider first the following exercise. We compute the amplitude for the scattering

$$\boxed{\pi(p_1) + \pi(p_2) \longrightarrow \pi(p_3) + \pi(p_4)}$$

with on-shell external particles. We have 3 diagrams which contribute to the amplitude.



We use the non linear field redefinition.

$$i\mathcal{M} = \frac{i}{v}(-2p_1\cdot p_2)\frac{i}{s-m_h^2}\frac{i}{v}(-2p_3\cdot p_4) + \frac{i}{v}(2p_1\cdot p_3)\frac{i}{t-m_h^2}\frac{i}{v}(2p_2\cdot p_4) +$$
$$+\frac{i}{v}(2p_1\cdot p_4)\frac{i}{u-m_h^2}\frac{i}{v}(2p_2\cdot p_3)$$

In this case:

$$\boxed{\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2 = 2p_1\cdot p_2 = 2p_3\cdot p_4 \\ t &= (p_1-p_3)^2 = (p_2-p_4)^2 = -2p_1 p_3 = -2p_2 p_4 \\ u &= (p_1-p_4)^2 = (p_2-p_3)^2 = -2p_1\cdot p_4 = -2p_2\cdot p_3 \end{aligned}}$$

$$i\mathcal{M} = \frac{(-i)}{v^2}\frac{s^2}{(s-m^2)} + \frac{(-i)}{v^2}\frac{t^2}{t-m_h^2} + \frac{(-i)}{v^2}\frac{u^2}{u-m_h^2}$$

$$= \frac{(-i)}{v^2}\frac{\left[s^2(t-m_h^2)(u-m_h^2) + t^2(s-m_h^2)(u-m_h^2) + u^2(s-m_h^2)(t-m_h^2)\right]}{(s-m_h^2)(t-m_h^2)(u-m_h^2)}$$

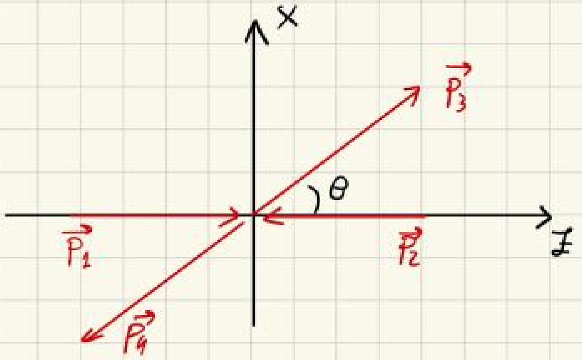
Consider the numerator (in the square bracket). We rewrite it as:

$$s^2tu + t^2su + u^2st + m_h^4(s^2+t^2+u^2) - m_h^2\left[s^2(t+u) + t^2(s+u) + u^2(s+t)\right] =$$

$$= \underbrace{stu(s+t+u)}_{\downarrow} - m_h^2\left[s^2(t+u) + t^2(s+u) + u^2(s+t)\right] + m_h^4(s^2+t^2+u^2)$$

Again we note that the "dangerous" term that would lead to an amplitude that grows with energy actually vanish in the sum since $s+t+u=0$.

In the C.O.M. frame we write:



$$p_1 = (E,0,0,E) \qquad p_3 = (E, E\sin\theta, 0, E\cos\theta)$$

$$p_2 = (E,0,0,-E) \qquad p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

$$\longrightarrow \begin{cases} s = 4E^2 \\ t = -2p_1\cdot p_3 = -2E^2(1-\cos\theta) \\ u = -2p_1\cdot p_4 = -2E^2(1+\cos\theta) \end{cases}$$

We consider a *fixed-angle scattering* and we suppose that the energy is much smaller compared to the mass m_h. We approximate the amplitude as follows:

$$i\mathcal{M} = \frac{(-i)}{v^2}\left[\frac{s^2}{(s-m_h^2)} + \frac{t^2}{(t-m_h^2)} + \frac{u^2}{(u-m_h^2)}\right] \qquad \text{(we neglect } s,t,u \text{ compared to } m_h^2)$$

$$= \frac{(-i)}{v^2}\left(-\frac{1}{m_h}\right)(s^2+t^2+u^2) = \frac{+i}{v^2 m_h^2}(s^2+t^2+u^2)$$

Using $m_h^2 = 2\mu^2 = 2\lambda v^2 \rightarrow \frac{1}{v^2} = \frac{2\lambda}{m_h^2}$:

$$\boxed{i\mathcal{M} = \frac{2i\lambda}{m_h^4}(s^2+t^2+u^2) \qquad s,t,u \ll m_h^2}$$

What Lagrangian describes this "low energy limit"?

We can try to construct such Lagrangian from "first principles"

i) The Lagrangian we seek must contain $\pi(x)$ as dynamical degree of freedom.

ii) We ask, therefore, the following question : " What is the most general Lagrangian which describes a Goldstone boson arising from the spontaneous breaking of an $SO(2) \cong U(1)$ simmetry?

iii) To answer this question, we apply our general criteria :
   a) The Goldstone boson is a real scalar field
   b) It enjoys the shift simmetry $\pi \longrightarrow \pi + \text{const}$

Let's write a Lagrangian consistent with these rules.

At the renormalizable level we only have one possibility :

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi)} \qquad \text{(Free theory of a scalar massless field)}$$

However, if we give up with the requirement of renormalizability, we can write additional terms :

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) + c\,\underline{\left[(\partial_\mu \pi)(\partial^\mu \pi)\right]^2}}$$

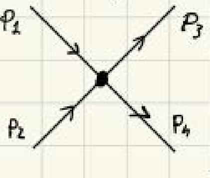
This term, consistent with our rules, involves 4 fields π and 4 derivatives. Therefore it has dimension 8 and must be multiplied by a constant "c" with mass dimension $[M]^{-4}$

We have classified these theories as non-renormalizable, an indeed they are. However, we now have the possibility to give a physical interpretation of these theories.

First of all let's compute the scattering amplitude of $\pi(P_1)\pi(P_2) \longrightarrow \pi(P_3)\pi(P_4)$ with the previous lagrangian.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) + \underbrace{c\,(\partial_\mu \pi)(\partial^\mu \pi)(\partial_\nu \pi)(\partial^\nu \pi)}_{\mathcal{L}_{int}}$$

the interaction part gives a total of $4! = 24$ contractions divided into 3 independent structures. Taking the limit of $E \ll m_n$ the propagator collapse into a contact interaction.



$$= ic\left[(-iP_{1\mu})(-iP_2^{\mu})(iP_{3\nu})(iP_4^{\nu})\times 8 + (-iP_{1\mu})(-iP_{2\nu})(iP_3^{\mu})(iP_4^{\nu})\times 8 + (-iP_{1\mu})(-iP_{2\nu})(iP_4^{\mu})(iP_3^{\nu})\times 8\right]$$

$$= ic\cdot 8\left[(P_1\cdot P_2)(P_3\cdot P_4) + (P_1\cdot P_3)(P_2\cdot P_4) + (P_1\cdot P_4)(P_2\cdot P_3)\right] =$$

$$= ic\cdot 2\left[(2P_1\cdot P_2)(2P_3\cdot P_4) + (2P_1\cdot P_3)(2P_2\cdot P_4) + (2P_1\cdot P_4)(2P_2\cdot P_3)\right] =$$

$$= 2ic\,(s^2+t^2+u^2)$$

We find the same amplitude if we identify $c \equiv \frac{\lambda}{m_n^4}$. We arrive at:

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) + \frac{\lambda}{m_n^4}\left[(\partial_\mu \pi)(\partial^\mu \pi)\right]^2}$$

In this Lagrangian, the massive mode does not appear explicitely : we say that it has been integrated out.

We call the above Lagrangian an *effective field theory* describing the dynamics of the Goldstone boson arising from the $SO(2)$ breaking.

$\longrightarrow$ Limit described by $\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) + \frac{\lambda}{m_h^4}\left[(\partial_\mu \pi)(\partial^\mu \pi)\right]^2$

i) Non-renormalizable theories can be considered to be low-energy effective field theories in which non-renormalizable operators arise because heavy states that do not partecipate to the dynamics have been integrated out.

ii) We now understand why, physics side, the non-renormalizable operators are characterized by coefficients with a negative mass dimension. This dimensionfull parameters keep track of heavy states integrated out. For instance, in our case, it is the mass of the radial mode.

iii) So far, we encountered two kinds of non-renormalizable theories

$$\begin{tikzcd}
& \boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \frac{1}{2}m_h^2\chi^2 + \frac{1}{2}(\partial_\mu \theta)(\partial^\mu \theta) - \frac{\lambda}{4}\left(\chi^4 + \theta^4 - 4v\chi^3 + 2\chi^2\theta^2 + 4v\chi\theta^2\right)} \arrow[dl] \arrow[dr] & \\
\boxed{\begin{aligned} \mathcal{L}_\pi = {} & \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}m_h^2 h^2 + \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) \\ & + \frac{1}{2}(h^2 + 2hv)(\partial_\mu \pi)(\partial^\mu \pi) - \lambda v h^3 - \frac{\lambda}{4}h^4 \end{aligned}} & & \boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi) + \frac{\lambda}{m_h^4}\left[(\partial_\mu \pi)(\partial^\mu \pi)\right]^2}
\end{tikzcd}$$

Non-renormalizable terms from non-linear field redefinition

Non renormalizable operators from integrating out heavy fields

We note that these 2 kind of non renormalizable theories differ :

- In the 1st case, the scattering amplitudes do not grow with energy (we verified that in our explicit computations)
- In the effective field theory case, scattering amplitudes do grow with energy. For example the scattering amplitude for $\pi\pi \to \pi\pi$ computed in the E.F.T. is:
  $i\mathcal{M} = \frac{2i\lambda}{m_h^4}(s^2+t^2+u^2)$ and it grows with energy ; therefore we cannot extrapolate its validity up to arbitrary high energy because it will lead to unitarity violation.
  This makes sense, since at energies $E \simeq m_h$ there is enough energy to make the radial mode dynamical. This is a generic result : we say that effective field theories break down at some energy scale (A.K.A. the "cutoff" of the E.F.T.)
  This energy scale is the scale at which the amplitude grows in energy at the point of causing violation of unitarity. Physically this scale signals the existence of heavy particles that become dynamical.

The exact computation is on lecture 36, 12-18 ; we can show that for $\pi\pi \to \pi\pi$ the energy cutoff of the E.F.T. is $E \lesssim \lambda^{1/4} v$

PARTIAL SYMMETRY BREAKING

We try to set a more general discussion. Consider a scalar theory with a global symmety G.

$$\boxed{\phi^a(x) \longrightarrow \phi'^a(x) = \phi^a(x) - i\alpha_A (T^A)^a{}_b \phi^b(x)} \qquad \text{infini}$$

Multiplet of scalar fields

$A = 1, 2, \dots, \dim G$

such that $[T^A, T^B] = i f_{ABC} T^C$

T^A are the generators of G. Their number is always equal to the dimension of the group. Their explicit form depends on the representation according to which the field ϕ^a transform.

The Lagrangian takes the form: $\mathcal{L} = \mathcal{L}_{KIN} - V(\vec{\phi})$; with the notation $\vec{\phi}$ to remind that we have a multiplet of scalar fields. The Lagrangian has to be invariant $D\mathcal{L} = 0$. This condition, if we focus on the potential term, can be written as follows:

$$DV(\vec{\phi}) = 0 \longrightarrow \frac{\partial V}{\partial \phi^a} D\phi^a = \frac{\partial V}{\partial \phi^a}(-i)\alpha_A (T^A)^a{}_b \phi^b = 0$$

that is:

$$\boxed{\frac{\partial V}{\partial \phi^a} \alpha_A (T^A)^a{}_b \phi^b = 0} \qquad \text{(Invariance of the potential under } G\text{)}$$

We minimize the energy of the theory. As usual the Kinetic part is minimized by looking for field configurations which are constant in time and homogeneous in space. As far as the potential is concerned, we suppose that the potential is minimized by some field configuration $\vec{\phi}_0$. We have 2 conditions:

$$\frac{\partial V}{\partial \phi^a}(\vec{\phi} = \vec{\phi}_0) = 0 \qquad \frac{\partial^2 V}{\partial \phi^a \partial \phi^b}(\vec{\phi} = \vec{\phi}_0) \text{ is positive semi-definite}$$

(eigenvalues ≥ 0)

We introduce shifted fields according to:

$$\boxed{\phi^a(x) \equiv \phi^a_0 + \chi^a(x)}$$

and consider the potential expanded around the minimum

$$V(\vec{\phi}) = V(\vec{\phi}_0 + \vec{\chi}) = V(\vec{\phi}_0) + \frac{\partial V}{\partial \phi^a}(\vec{\phi}_0)\chi^a + \frac{1}{2}\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}(\vec{\phi}_0)\chi^a \chi^b + \dots =$$

$$= V(\vec{\phi}_0) + \frac{1}{2}\underbrace{\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}(\vec{\phi}_0)}_{\text{"Mass Matrix"}}\chi^a \chi^b + \dots$$

The terms quadratic in the fields determine the "mass matrix" of the theory.

Example

Consider the case of $G \equiv SO(2)$ we discussed before with

$$\vec{\phi} \equiv \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \quad ; \quad \vec{\phi}_0 \equiv \begin{pmatrix} v \\ 0 \end{pmatrix} \quad ; \quad V(\phi_1,\phi_2) = \frac{-\mu^2}{2}(\phi_1^2+\phi_2^2) + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2$$

In this case we have:

$$\boxed{\frac{\partial^2 V}{\partial\phi_i\,\partial\phi_j}(\vec{\phi}_0) = \begin{pmatrix} 3v^2\lambda-\mu^2 & 0 \\ 0 & v^2\lambda-\mu^2 \end{pmatrix} = \begin{pmatrix} 2\mu^2 & 0 \\ 0 & 0 \end{pmatrix}}$$

(The eigenvalues represent the mass spectrum of the theory)

N.B. The mass matrix in general is not diagonal! For instance consider a different choice of vacuum given by:

$$\langle\phi_1\rangle = \frac{1}{\sqrt{2}}v \quad ; \quad \langle\phi_2\rangle = \frac{1}{\sqrt{2}}v \quad ; \quad \langle\phi_1\rangle^2 + \langle\phi_2\rangle^2 = v^2$$

we find:

$$\boxed{\frac{\partial^2 V}{\partial\phi_i\,\partial\phi_j}(\vec{\phi}_0) = \begin{pmatrix} 2v^2\lambda-\mu^2 & v^2\lambda \\ v^2\lambda & 2v^2\lambda-\mu^2 \end{pmatrix} = \begin{pmatrix} \mu^2 & \mu^2 \\ \mu^2 & \mu^2 \end{pmatrix}}$$

(The matrix is non-diagonal but the physics is unchanged. To read the mass spectrum we diagonalize the mass matrix: the eigenvalues are indeed $2\mu^2$ and 0!)

The matrix is non diagonal means that if we considered the shifted fields:

$$\begin{cases} \phi_1(x) = \frac{1}{\sqrt{2}}v + \chi_1(x) \\ \phi_2(x) = \frac{1}{\sqrt{2}}v + \chi_2(x) \end{cases}$$

the free Lagrangian reads:

$$\frac{1}{2}(\partial_\mu\chi_1)(\partial^\mu\chi_1) + \frac{1}{2}(\partial_\mu\chi_2)(\partial^\mu\chi_2) - \frac{1}{2}\mu^2(\chi_1^2+\chi_2^2+\underbrace{2\chi_1\cdot\chi_2})$$

we cannot read the mass spectrum because of the mixing term

The mass-spectrum however becomes transparent after diagonalization.

We now ask what is the action of the symmetry group on the vacuum configuration

WIGNER-WAYL SITUATION : the vacuum is left invariant by the action of the group

$$\longrightarrow \boxed{D(g)\vec{\phi}_0 = \vec{\phi}_0 \quad \forall g \in G \longrightarrow T^A\vec{\phi}_0 = \vec{0} \ , \ A=1,\dots,\dim(G)}$$

(PARTIAL) SPONTANEOUS SYMMETRY BREAKING : it takes place if the vacuum is not invariant under the action of G

$$\longrightarrow \boxed{T^A\vec{\phi}_0 \neq \vec{0}}$$

(T^A does not annihilate the vacuum)

GENERAL SITUATION : some of the generators annihilate the vacuum and some of the generators do not annihilate the vacuum

- If $T^a\vec{\phi}_0 = 0 \longrightarrow T^a$ is called UNBROKEN GENERATOR
- If $T^a\vec{\phi}_0 \neq 0 \longrightarrow T^a$ is called BROKEN GENERATOR

Example :

Consider the case of $G \equiv SO(3)$.

$$\vec{\Phi} \equiv \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} \qquad \mathcal{L} = \frac{1}{2}(\partial_\mu \phi_a)(\partial^\mu \phi_a) - V(\vec{\Phi}) \qquad V(\vec{\Phi}) = -\frac{\mu^2}{2}\sum_{a=1}^{3}\phi_a^2 + \frac{\lambda}{4}\left(\sum_{a=1}^{3}\phi_a^2\right)^2$$

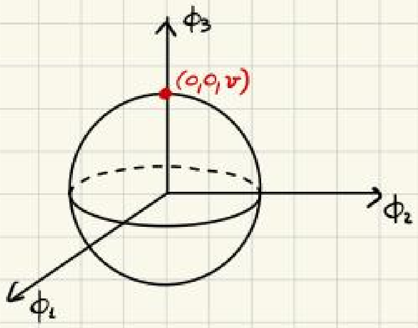
Defining $X \equiv (\phi_1^2+\phi_2^2+\phi_3^2)^{1/2}$, $V(X) = \frac{-\mu^2}{2}X^2 + \frac{\lambda}{4}X^4$; $\mu^2>0$, $\lambda>0$ the minimum is:

$$\boxed{\phi_1^2+\phi_2^2+\phi_3^2 = \frac{\mu^2}{\lambda} \equiv v^2}$$

Hence in this case we have a degeneracy of vacua described by a 2 dim. sphere of radius v.
All degenerate vacua are related by an SO(3) transformation

We now make a choice, for instance :

$$\boxed{\langle\vec{\Phi}\rangle \equiv \vec{\Phi}_0 = \begin{pmatrix}0\\0\\v\end{pmatrix}}$$



This specific choice of the vacuum breaks the original SO(3) symmetry. However there is a subgroup of SO(3) under which the vacuum is invariant : the vacuum indeed is invariant under SO(2) rotations around the third axis (in this case). Therefore :

$$T^1 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \longrightarrow T^1\vec{\Phi}_0 = \begin{pmatrix}0\\-iv\\0\end{pmatrix} \neq \vec{0} \longrightarrow \text{BROKEN}$$

$$T^2 = \begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix} \longrightarrow T^2\vec{\Phi}_0 = \begin{pmatrix}iv\\0\\0\end{pmatrix} \neq \vec{0} \longrightarrow \text{BROKEN}$$

$$T^3 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \longrightarrow T^3\vec{\Phi}_0 = \begin{pmatrix}0\\0\\0\end{pmatrix} = \vec{0} \longrightarrow \text{UNBROKEN}$$

The following result holds:

- The unbroken generators T^a close a subalgebra. The elements $\exp(-i\alpha_a T^a)$ form a subgroup usually denoted with H . (In this case we say that the symmetry is broken $G \longrightarrow H$) .

Proof :

Let's consider the commutator $[T^a, T^b]$. We write : $[T^a, T^b] = i f_{abc} T^c + i f_{ab\hat{c}} T^{\hat{c}}$

we apply to both sides the vacuum: $\underbrace{[T^a, T^b]\vec{\Phi}_0}_{=0} = i f_{abc}\underbrace{T^c\vec{\Phi}_0}_{=0} + i f_{ab\hat{c}}\underbrace{T^{\hat{c}}\vec{\Phi}_0}_{\neq 0}$

so we find that $f_{ab\hat{c}} T^{\hat{c}}\vec{\Phi}_0 = 0 \longrightarrow \boxed{f_{ab\hat{c}} = 0}$ $\forall\, a, b, \hat{c}$

Therefore $\boxed{[T^a, T^b] = i f_{abc} T^c}$ (closed subalgebra) ☐

We now come back to the equation $\frac{\partial V}{\partial \phi^a}\alpha_A (T^A)^a{}_b \phi^b = 0$ and differentiate it with respect to ϕ^c:

$$\frac{\partial^2 V}{\partial \phi^a \partial \phi^c}\alpha_A (T^A)^a{}_b \phi^b + \frac{\partial V}{\partial \phi^a}\alpha_A (T^A)^a{}_b \delta_{bc} = 0$$

$$\alpha_A \left[\frac{\partial^2 V}{\partial \phi^a \partial \phi^c}(T^A)^a{}_b \phi^b + \frac{\partial V}{\partial \phi^a}(T^A)^a{}_c\right] = 0$$

$$\frac{\partial^2 V}{\partial \phi^a \partial \phi^c}(T^A)^a{}_b \phi^b + \frac{\partial V}{\partial \phi^a}(T^A)^a{}_c = 0$$

We evaluate for $\vec{\phi}_0$:

$$\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}(\vec{\phi}_0)(T^A)^a{}_b \phi_0^b + \cancel{\frac{\partial V}{\partial \phi^a}(\vec{\phi}_0)}(T^A)^a{}_c = 0 \longrightarrow \boxed{\underbrace{\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}}_{\text{This is the mass matrix we defined before}}(\vec{\phi}_0)(T^A\vec{\phi}_0)^a = 0} \quad A = 1 \dots \dim(G)$$

$\overset{!!}{0}$

There are 2 cases:

i) The case in which "A" corresponds to an unbroken generator $T^A \equiv T^a$; $T^a\vec{\phi}_0 = \vec{0}$ (the above condition is trivially verified).

ii) The case in which "A" corresponds to a broken generator $T^A \equiv T^{\hat{a}}$; $T^{\hat{a}}\vec{\phi}_0 \neq \vec{0}$
In this case we extract non-trivial informations:

$$\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}(\vec{\phi}_0)\underbrace{(T^A\vec{\phi}_0)^a}_{\neq \vec{0}} = 0 \longrightarrow$$ $T^{\hat{a}}\vec{\phi}_0$ is an eigenvector of the mass matrix with zero eigenvalue

Since we can repeat this reasoning for all $\hat{a}$ independently, we conclude that:

There is one Goldstone boson (massless scalar) for every linearly independent spontaneous broken symmetry generator:

$$\# \text{ Goldstone bosons} = \dim G - \dim H$$

So there is a 1 to 1 correspondence between the Goldstone fields and the broken generators $T^{\hat{a}}$.

Comment:

We have separated the generators T^A into $\{T^A\} = \{T^a, T^{\hat{a}}\} = \{\underbrace{T^1, \dots, T^{\dim H}}_{\text{form a subalgebra that generates the subgroup } H}; T^{\hat{a}}\}$

In general the subgroup H is not a normal subgrop. In fact we have:

$$[T^a, T^{\hat{b}}] = i f_{a\hat{b}c}T^c + i f_{a\hat{b}\hat{c}}T^{\hat{c}} = -i f_{ab\hat{c}}T^c + i f_{a\hat{b}\hat{c}}T^{\hat{c}} = i f_{a\hat{b}\hat{c}}T^{\hat{c}}$$ which is not h

As a final point we would like to find a generalization of the non linear parametrization that describes the Goldstone fields.
As we have seen, there is a one-to-one correspondence between Goldstone bosons and the broken generators. This suggests us to consider the exponential matrix:

$$\boxed{\mathcal{U}(x) \equiv \exp\left(\frac{i\pi^{\hat{a}}2T^{\hat{a}}}{f_\pi}\right)}$$ Goldstone boson matrix

with $\pi^{\hat{a}}(x)$ $\hat{a} = 1, \dots, \dim G - \dim H$ the goldstone fields.

We discuss the case $\boxed{SO(N) \longrightarrow SO(N-1)}$

Consider a multiplet of real scalar fields $\vec{\Phi} = \begin{pmatrix} \phi^1 \\ \vdots \\ \phi^N \end{pmatrix}$ with Lagrangian density:

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\Phi} \cdot \partial^\mu \vec{\Phi} + \frac{\mu^2}{2}\vec{\Phi}\cdot\vec{\Phi} - \frac{\lambda}{4}(\vec{\Phi}\cdot\vec{\Phi})^2}$$

which is invariant under the field transformation

$$\boxed{\vec{\Phi}' \longrightarrow \exp(-i\alpha_A T^A)\vec{\Phi}}$$

T^A are the $\frac{N^2-N}{2}$ generators of $SO(N)$ in the fundamental (defining) representation

Minima are defined by the condition:

$$\boxed{\vec{\Phi}\cdot\vec{\Phi} = \frac{\mu^2}{\lambda} \equiv v^2}$$

We choose the vacuum:

$$\boxed{\langle\vec{\Phi}\rangle = \begin{pmatrix} 0 \\ \vdots \\ v \end{pmatrix}}$$

The generators of $SO(N)$ in the defining representations are given as follows. The generators of the algebra $so(N)$ are symmetric matrices. They can be organized as follows:

$$T^A = \left\{ \underbrace{T^a = \left(\begin{array}{c|c} t^a & \vec{0} \\ \hline \vec{0}^T & 0 \end{array}\right)}_{\text{it generates the } SO(N-1) \text{ subgroup}} ; \underbrace{\frac{1}{2}\left(\begin{array}{c|c} 0 & \begin{matrix} -i \\ 0 \\ \vdots \end{matrix} \\ \hline i\,0 \cdots 0 & 0 \end{array}\right) ; \frac{1}{2}\left(\begin{array}{c|c} 0 & \begin{matrix} 0 \\ \vdots \\ -i \end{matrix} \\ \hline 0 \cdots i & 0 \end{array}\right)}_{N-1 \text{ broken generators } T^{\hat{a}}} \right\}$$

We write:

$$\vec{\Phi}(x) = \underbrace{\exp\left(i\overset{N-1 \text{ real scalar fields}}{\pi^{\hat{a}}(x)}\frac{2T^{\hat{a}}}{v}\right)}_{\text{Goldstone matrix } \mathcal{U}(\pi^{\hat{a}}(x))}\begin{pmatrix} \vec{0} \\ \underset{\text{scalar field}}{h(x)} + v \end{pmatrix}$$

We rewrite the Lagrangian in terms of $\vec{\Phi}(x)$. The potential is trivial, since the Goldstone matrix drops. In fact we write $\vec{\Phi}\cdot\vec{\Phi}$ as:

$$\vec{\Phi}\cdot\vec{\Phi} = \vec{\Phi}^T\vec{\Phi} = (\vec{0}^T, h(x)+v)\,\mathcal{U}(\pi^{\hat{a}}(x))^T\,\mathcal{U}(\pi^{\hat{a}}(x))\begin{pmatrix} \vec{0} \\ h+v \end{pmatrix} = (h(x)+v)^2$$

and the potential is:

$$\boxed{V = \frac{-\mu^2}{2}(h+v)^2 + \frac{\lambda}{4}(h+v)^4}$$

The field $\pi^{\hat{a}}(x)$, therefore, correctly describes the Goldstone fields. Furthermore, we can now check <u>how the Goldstone field transform.</u>

i) First, we consider transformations generated by the broken symmetry.

$$\phi(x) = \exp\left(-i\vec{\alpha}_{\hat a} T^{\hat a}\right) \vec{\Phi}(x) = \exp\left(-i\alpha_{\hat a} T^{\hat a}\right) \exp\left(i\pi^{\hat a}(x) \cdot \frac{2\hat{T}^{a}}{v}\right) \begin{pmatrix} \vec{0} \\ h(x)+v \end{pmatrix}$$

Using that formula and focusing on infinitesiomal transformation we get:

$$\boxed{\pi^{\hat a}(x) \longrightarrow \pi^{\hat a}(x) - \frac{v}{2}\alpha_{\hat a}}$$

(Non linear shift under the broken symmetry)

ii) Consider now transformations along the unbroken direction.

$$\vec{\Phi}(x) \longrightarrow \exp\left(-i\alpha_a T^a\right) \vec{\Phi}(x) =$$

$$= \exp\left(-i\alpha_a T^a\right) \exp\left(i\pi^{\hat a}(x) \frac{2T^{\hat a}}{v}\right) \exp\left(i\alpha_b T^b\right) \begin{pmatrix} \vec{0} \\ h+v \end{pmatrix} =$$

expanding both in α_a and $\pi^{\hat a}$ we get:

$$\cong \left(\mathbb{1} - i\alpha_a T^a\right)\left(\mathbb{1} + i\pi^{\hat a}(x)\frac{2T^{\hat a}}{v}\right)\left(\mathbb{1} + i\alpha_b T^b\right) =$$

$$= \mathbb{1} + i\pi^{\hat a}\frac{2T^{\hat a}}{v} - i\alpha_a T^a + i\alpha_a T^a + \alpha_a \pi^{\hat a}\frac{2}{v} T^a T^{\hat a} - \alpha_a \frac{\pi^{\hat a} 2 T^{\hat a}}{v} T^a =$$

$$= \mathbb{1} + i\frac{\pi^{\hat a} 2 T^a}{v} + \frac{2\pi^{\hat a}\alpha_a}{v}\left[T^a, T^{\hat a}\right] =$$

$$= \mathbb{1} + i\frac{\pi^{\hat a} 2 T^a}{v} + \frac{2\pi^{\hat a}\alpha_a}{v} i f_{a\hat a \hat b} T^{\hat b} =$$

$$= \mathbb{1} + \frac{2i}{v}\left(\pi^{\hat a} + \alpha_a f_{a\hat b \hat a}\pi^{\hat b}\right) T^{\hat a}$$

From the comparison we get:

$$\boxed{\pi^{\hat a}(x) \xrightarrow{H} \pi^{\hat a}(x) - i\alpha_a i f_{a\hat b \hat a}\pi^{\hat b}(x)}$$

(The Goldstone boson transforms linearly under H)

The unbroken group H is indeed realized à la Wigner Wayl, and all fields, including the Goldstone must transform linearly under its action.

From the infinitesimal transformation we can extract the generators in the representation of H according to which the Goldstone fields transform.

$$\pi^{\hat a}(x) \longrightarrow \pi'^{\hat a}(x) - i\alpha_a (t^a)_{\hat a \hat b}\pi^{\hat b}(x)$$

(a runs over the unbroken generators)

$$\longrightarrow \boxed{(t^a)_{\hat a \hat b} \equiv i f_{a\hat b \hat a}}$$

The generators are written in terms of the structure constant of the group G, with "a" that runs over the unbroken generators and "$\hat a$" and "$\hat b$" running over the broken part $\hat a, \hat b = 1, \dots, \dim G - \dim H = 1, \dots, \#G.B.$

Remember the definition of the adjoint representation of a Lie Algebra:

$$\mathrm{ad}: g \longrightarrow g(g) \qquad (g(g) \text{ space of all endomorphism from the algebra to itself})$$

$$(\mathrm{ad}(x))y = [x, y] \quad \forall x, y \in g$$

We extract the matrix elements which define the operator $\mathrm{ad}(T^A)$ in the basis defined by the generators: $\boxed{(t^A)_{BC} = i f_{ABC}.}$

Consider the adjoint of the group and consider its restriction to the subgroup H. We compute $(\mathrm{ad}(T^a))(T^B)$

- $(\mathrm{ad}(T^a))T^b = [T^a, T^b] = i f_{abc} T^c$ ⟶ this is the adjoint of the subg. H
- $(\mathrm{ad}(T^a))T^{\hat{b}} = [T^a, T^{\hat{b}}] = i f_{a\hat{b}\hat{c}} T^{\hat{c}}$ ⟶ this is the subspace of $T^{\hat{a}}$ left invariant by the action of $\mathrm{ad}(T^a)$

The bottom line is that the adjoint representation of G, when restricted to the unbroken subgroup H, decomposes into the direct sum of 2 representations. The 1st one is simply the adjoint of H, while the 2nd one is the representation according to which the broken generators transform (under the unbroken symmetry)

$$\mathrm{Ad}(g) \overset{h}{\sim} \underset{\substack{\uparrow \\ \text{with generators} \\ (t^a)_{bc} = i f_{acb}}}{\mathrm{Ad}(h)} \oplus \underset{\substack{\nwarrow \\ \text{with generators} \\ (t^a)_{\hat{b}\hat{c}} \equiv i f_{a\hat{c}\hat{b}}}}{R_\pi}$$

In few words: the Goldstone particles form the same type of multiplet under the unbroken symmetry as the generators $T^{\hat{a}}$ of the broken symmetry.

TOWARDS THE GOLDSTONE THEOREM

We discuss the Goldstone theorem in the full quantum theory. Consider a QFT with a global symmetry group (continuous) G. The symmetry group is realized by means of the action of unitary operators on the states of the theory, and the corresponding hermitian generators are indicated with Q^A.
We have hence that $[Q^A, H] = 0$ (Internal symmetry). Remember that the ground state is the one with the lowest energy $H|\Omega\rangle = E_0|\Omega\rangle$. (with E_0 tipically normalized to zero).

If $Q^A|\Omega\rangle = 0$, the symmetry is realized à la Wigner-Weyl

As we discussed, spontaneous symmetry breaking is realized if the ground state is not invariant under the action of a symmetry and its trademark is a degeneracy in the vacua. It is then natural to interpret the spontaneous symmetry breaking case as follows:

If $Q^A|\Omega\rangle \neq 0$, the symmetry is realized à la Nambu-Goldstone

This seems to be a good "quantum definition" since:

$$H Q^A |\Omega\rangle = H Q^A |\Omega\rangle = Q^A E_0 |\Omega\rangle = E_0 (Q^A |\Omega\rangle)$$

so that $Q^A|\Omega\rangle \neq 0$ has the same energy of $|\Omega\rangle$. Precisely what we observed in the classical analysis.

There is, however, a problem with this definition known as the Fabri-Picasso theorem. The problem is that if we apply $Q^A|\Omega\rangle$, either we get zero or the state $Q^A|\Omega\rangle$ is not well defined in the Hilbert space.

Consider the norm $\| Q^A|\Omega\rangle\|^2$:

$$\begin{aligned} \| Q^A|\Omega\rangle \|^2 &= \langle\Omega| Q^A Q^A|\Omega\rangle \qquad ; \quad Q^A = \int d^3\vec{x}\, J^0_A(\vec{x},t) \\ &= \int d^3\vec{x} \langle\Omega| Q^A J^0_A(x)|\Omega\rangle \qquad ; \quad J^0_A = e^{ix\cdot P} J^0_A(0)\, e^{-ix\cdot P} \\ &= \int d^3\vec{x} \langle\Omega| Q^A e^{ix\cdot P} J^0_A(0) e^{-ix\cdot P}|\Omega\rangle \overset{\curvearrowleft}{=} (Q^A \text{ and } P^\mu \text{ commute and } e^{-ix\cdot P}|\Omega\rangle = |\Omega\rangle) \\ &= \int d^3\vec{x} \langle\Omega| Q^A J^0_A(0)|\Omega\rangle \longrightarrow \infty \quad \text{unless } Q^A|\Omega\rangle = 0 \end{aligned}$$

We need therefore a different definition. The key observations are the following.

i) Consider the commutator $[Q^A, O(y)]$ where $O(y)$ is some operator (made out of fields). We write:

$$[Q^A, O(y)] = \int d^3\vec{x}\, \underbrace{[J^0_A(x), O(y)]}_{=0} \qquad \text{If } (x-y)^2 < 0 \rightarrow (x^0-y^0)^2 - |\vec{x}-\vec{y}|^2 < 0 \rightarrow |\vec{x}-\vec{y}|^2 > (x^0-y^0)^2$$

The above object is well defined because it receives contributions only from a finite volume $|\vec{x}-\vec{y}|^2 < (x^0-y^0)^2$. The 1st insight, therefore is that it is possible to overcome the obstruction of the Fabri-Picasso theorem by considering commutators involving Q^A.

ii) We now need an alternative condition to $Q^A|\Omega\rangle \neq 0$. Consider the transformation property of a quantum field under the action of G. We wrote:

$$\boxed{[Q^A, O^a(x)] = -(T^A)^a{}_b\, O^b(x)} \quad \bigstar$$

Consider the case in which G is realized à la Wigner Weyl. In such case $Q^A|\Omega\rangle = 0$. So:

$$\underbrace{\langle\Omega| [Q^A, O^a(x)] |\Omega\rangle}_{=0 \text{ If the symm. is realized à la Wigner Weyl}} = -(T^A)^a{}_b \underbrace{\langle\Omega| O^b |\Omega\rangle}_{\text{this is a true vacuum expectation value}} \overset{!}{=} 0$$

★ is the infinitesimal form of the transformation property:

$$\boxed{O^a(x) \longrightarrow O'^a(x) = \mathcal{U}^\dagger(\vec{\alpha})\, O^a(x)\, \mathcal{U}(\vec{\alpha}) = D(\vec{\alpha})^a{}_b\, O^b(x)}$$

Suppose that the operator $O^a(x)$ has a non-zero VEV $\langle\Omega| O^a(x)|\Omega\rangle \neq 0$, then it will transform according to:

$$\langle\Omega| O^a(x)|\Omega\rangle \longrightarrow \langle\Omega| O'^a(x)|\Omega\rangle = \underbrace{D(\vec{\alpha})^a{}_b}_{\exp(-i\alpha_A T^A)^a_b} \langle\Omega| O^b(x)|\Omega\rangle = \quad \langle\Omega| O^a(x)|\Omega\rangle$$

We learn something interesting. Only operators which are invariant under G may acquire a non-zero VEV if G is realized à la Wigner Weyl. This is important because in our setup space-time symmetries are always realized à la Wigner Weyl. Therefore only operators which are Lorentz scalars may acquire a non-vanishing VEV. For this reason we tipically consider $O(x)$ to be an elementary scalar field even though $O(x)$ could also be a scalar composite operator.

We now put together i) and ii). Since for a symmetry realized à la Wigner Weyl we have $<\Omega|[Q^A, O^a(x)]|\Omega>$, it seems natural to require that a trademark for a spontaneous broken symmetry is the condition:

$$\boxed{<\Omega|[Q^A, O^a(x)]|\Omega> = -(T^A)^a{}_b <\Omega|O^b(x)|\Omega> \neq 0}$$

$O^a(x)$ are operators that transform non trivially under the group G but they are Lorentz scalars. Because of i) the VEV is well defined. We can try to be even more specific. Consider a scalar operator $O^a(x)$ with non trivial transformation properties under G. In particular suppose that this operator transforms according to some irreps of G. A trade mark of spontaneous symm. breaking is the presence of a <u>non-vanishing order parameter</u>:

$$\boxed{<\Omega|O^b(x)|\Omega> \equiv v^b \neq 0} \quad \text{(At least for some component } b\text{)}$$

In fact the presence of a non-vanishing order parameter implies that:

$$\boxed{<\Omega|[Q^A, O^a(x)]|\Omega> \neq 0} \quad \text{(At least for one "A" and one "a")}$$

In fact:

$$<\Omega|[Q^A, O^a(x)]|\Omega> = -(T^A)^a{}_c <\Omega|O^c(x)|\Omega> =$$

$$= -(T^A)^a{}_c \, \delta^c{}_b \, v^b =$$

$$= -(T^A)^a{}_b \, v^b \neq 0 \quad \text{Since } v^b \neq 0$$

More precisely there must be at least one "A" and one "a" such that $-(T^A)^a{}_b v^b \neq 0$, otherwise $\vec{v}$ would define an invariant subspace (that is not possible since the representation is irreducible).

Example :

Consider the case $G = SO(2)$ we discussed at the beginning. The vector $\vec{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is a doublet of scalar fields that transform according to the fundamental representation of G.

$$\boxed{\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow R(\theta) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \simeq (\mathbb{1} - i\theta T) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad , \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

Since $D\phi_1 = -\phi_2$ and $D\phi_2 = +\phi_1$ then the current and hence the charge are:

$$J^\mu = (\partial^\mu \phi_2)\phi_1 - (\partial^\mu \phi_1)\phi_2 \longrightarrow \boxed{Q = \int d^3\vec{x} \left[(\partial_t \phi_2)\phi_1 - (\partial_t \phi_1)\phi_2\right]}$$

We compute the commutator :

$$\boxed{[Q^A, \phi_m(x)] = -(T^A)_{nm}\phi_m(x) \longrightarrow [Q, \vec{\Phi}(x)] = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} i\phi_2 \\ -i\phi_1 \end{pmatrix} \longrightarrow \begin{cases} [Q, \phi_1] = i\phi_2 \\ [Q, \phi_2] = -i\phi_1 \end{cases}}$$

The symmetry is spontaneously broken if there is a non-zero order parameter. If we set

$$\boxed{\langle \phi_1 \rangle = v \longrightarrow \langle [Q, \phi_2] \rangle = 0}$$

We see explicitely that a non-zero order parameter implies a non-zero commutator of Q, previously stated.

GOLDSTONE THEOREM

For a given continuous global symmetry, if $\langle \Omega | [Q^A, O^a(x)] | \Omega \rangle \neq 0$ at least for one "A" and one "a", then there is a zero mass particle in the theory.

Proof:

We consider the quantity $\langle \Omega | J^\mu_A(y) O^a(x) | \Omega \rangle$. We elaborate as follows: we introduce a resolution of the identity in the form:

$$\boxed{\mathbb{1} = \sum_N |N\rangle\langle N|}$$

(The sum extends to the vacuum, to single particle states and multi-particle states).

We don't need to construct explicitely such states, we only need the information that these are eigenstates of P^μ with total energy and total momentum:

$$\boxed{H|N\rangle = E_N|N\rangle \quad ; \quad \vec{P}|N\rangle = \vec{P}_N|N\rangle}$$

Therefore:

$$\langle \Omega | J^\mu_A(y) O^a(x) | \Omega \rangle = \sum_N \langle \Omega | J^\mu_A(y) | N \rangle \langle N | O^a(x) | \Omega \rangle$$

We now write:

$$\begin{cases} J^\mu_A(y) = e^{iy\cdot P} J^\mu_A(0) e^{-iy\cdot P} \\ O^a(x) = e^{ix P} O^a(0) e^{-ix\cdot P} \end{cases}$$

and then:

$$= \sum_N \langle \Omega | e^{iy\cdot P} J^\mu_A(0) e^{-iy\cdot P} |N\rangle\langle N| e^{ix\cdot P} O^a(0) e^{-ix\cdot P} |\Omega\rangle \overset{\text{using the vacuum invariance}}{=}$$

$$= \sum_N \langle \Omega | J^\mu_A(0) e^{-iy\cdot P_N} |N\rangle\langle N| e^{ix P_N} O^a(0) |\Omega\rangle =$$

$$= \sum_N e^{i(x-y)\cdot P_N} \langle \Omega | J^\mu_A(0) |N\rangle\langle N| O^a(0) |\Omega\rangle =$$

$$= \int d^4\phi \sum_N \langle \Omega | J^\mu_A(0) |N\rangle\langle N| O^a(0) |\Omega\rangle e^{i(x-y)\cdot P} \delta_{P-P_N}$$

We now define:

$$\boxed{\frac{1}{(2\pi)^3} \rho^\mu_{A,n}(P) \equiv \sum_N \langle \Omega | J^\mu_A(0) |N\rangle\langle N| O^a(0) |\Omega\rangle \delta_{P-P_N}} \quad \bigstar$$

We rewrite $\rho^{\mu}_{A,n}(P)$ as: $\boxed{\rho^{\mu}_{A,n}(P) = P^{\mu}\rho_{A,n}(P^2)\,\theta(P^0)}$ (using just Lorentz covariance)

We are now going to prove the <u>contrapositive proposition</u> (which is logically equivalent)

If, except for the vacuum, all states are such that $P_\mu P^\mu \geq \varepsilon > 0$ (No massless states) then $\langle\Omega|[Q^A, O^a(x)]|\Omega\rangle = 0$.

Two considerations:

a) Consider some value of P^μ such that $p^2 < \varepsilon$. Then the R.H.S. of the equation ★ is zero. Infact in the sum the vacuum state doesn't contribute since $\langle\Omega|J^{\mu}_{A}(0)|\Omega\rangle = 0$ and all other states have $P^2 \geq \varepsilon$. So we have

$$\boxed{\rho_{A,n}(P^2) = 0 \ ; \ P^2 \leq \varepsilon}$$

b) Consider now

$$\langle\Omega|J^{\mu}_{A}(y)\,O^a(x)|\Omega\rangle = \frac{1}{(2\pi)^3}\int d^4p\; p^\mu \rho_{A,n}(P^2)\,\theta(P^0)\,e^{i(x-y)\cdot P}$$

If we take $\frac{\partial}{\partial y^\mu}$ and use the conservation of the current

$$0 \overset{!}{=} \frac{1}{(2\pi)^3}\int d^4p\; p^\mu \rho_{A,n}(P^2)\,\theta(P^0)\,e^{i(x-y)\cdot P}$$

and if we consider the case $p^2 \geq \varepsilon > 0$ it follows that:

$$\boxed{\rho_{A,n}(P^2) = 0 \ , \ \ P^2 \geq \varepsilon}$$

We conclude that $\langle\Omega|J^{\mu}_{A}(y)\,O^a(x)|\Omega\rangle = 0$. Similarly, since the proof does not depend on the order, we also have $\langle\Omega|O^a(x)\,J^{\mu}_{A}(y)|\Omega\rangle = 0$. Consequently:

$$\boxed{\langle\Omega|[J^{\mu}_{A}(y), O^a(x)]|\Omega\rangle = 0}$$

and since $\langle\Omega|[Q^A, O^a(x)]|\Omega\rangle = \int d^3\vec{y}\; \underbrace{\langle\Omega|[J^0(y), O^a(x)]|\Omega\rangle}_{=0}$, we have that:

$$\boxed{\langle\Omega|[Q^A, O^a(x)]|\Omega\rangle = 0}$$

□

<u>Comment</u>:

Notice that the proof is invalidated if there are massless particles in the theory. The equation $P^2\rho_{A,n}(P^2) = 0$ must hold, since it is a consequence of current conservation. However it is satisfied if we take $\rho_{A,n}(P^2) = \delta(P^2)$. Consequently:

$$\frac{1}{(2\pi)^3}\,P^\mu \rho_{A,n}(P^2)\,\theta(P^0) = \frac{1}{(2\pi)^3}\,P^\mu\,\delta(P^2)\,\theta(P^0) = \frac{1}{(2\pi)^3}\,P^\mu \frac{1}{2P^0}\,\delta(P^0 - |\vec{P}|)$$

it means that in the sum over $|N\rangle$ there is a massless single particle state since it would contribute as:

$$\int \frac{d^3\vec{K}}{(2\pi)^3 2K^0}\langle\Omega|J^{\mu}_{A}(0)|\vec{K}\rangle\langle\vec{K}|O^a(0)|\Omega\rangle\,\delta(P^0 - |\vec{K}|)\,\delta(\vec{P}-\vec{K}) = \frac{1}{(2\pi)^3}\langle\Omega|J^{\mu}_{A}(0)|\vec{P}\rangle\langle\vec{P}|O^a(0)|\Omega\rangle\,\frac{\delta(P^0 - |\vec{P}|)}{2P^0}$$

THE HIGGS MECHANISM

• ABELIAN GAUGE THEORY CASE

Consider the U(1) Gauge theory (Scalar QED)

$$\boxed{\begin{array}{l} \mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ D_\mu\phi \equiv (\partial_\mu + igA_\mu)\phi \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad\qquad \mu^2>0,\ \lambda>0 \end{array}}$$

the Lagrangian is invariant under the Gauge transformation

$$\boxed{\begin{array}{l} \phi(x) \longrightarrow \phi'(x) = \exp(-i\alpha(x))\,\phi(x) \\ A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{g}(\partial_\mu\alpha(x)) \end{array}}$$

This means that, written in terms of $\phi'(x)$ and $A'_\mu(x)$, the Lagrangian has precisely the same form. Consider for instance the Kinetic term:

$$\begin{aligned} D_\mu\phi = \partial_\mu\phi + igA_\mu\phi &= \partial_\mu(e^{i\alpha}\phi') + ig\left(A'_\mu - \frac{1}{g}(\partial_\mu\alpha)e^{i\alpha}\phi'\right. = \\ &= \cancel{i(\partial_\mu\alpha)e^{i\alpha}\phi'} + e^{i\alpha}(\partial_\mu\phi') + igA'_\mu e^{i\alpha}\phi' - \cancel{i(\partial_\mu\alpha)e^{i\alpha}\phi'} = \\ &= e^{i\alpha}(\partial_\mu\phi' + igA'_\mu\phi') = e^{i\alpha}D'_\mu\phi' \end{aligned}$$

$$\longrightarrow \quad \mathcal{L}_{kin} = (D_\mu\phi)^\dagger(D_\mu\phi) = (D_\mu\phi')^\dagger(D_\mu\phi') = \mathcal{L}'_{kin}$$

The potential gives rise to spontaneous symmetry breaking. This is the discussion we already had in the spontaneous symmetry breaking of SO(2). Classicaly, the vacua we defined by the condition $|\phi|^2 = \frac{v^2}{2\lambda}$. We make a specific choice of vacuum whi corresponds in cartesian field variables $\phi(x) \equiv \frac{1}{\sqrt{2}}(\phi_1(x) + i\,\phi_2(x))$, to: $<\phi_1> = v$, $<\phi_2> = 0$.

We introduce the shifted fields:

$$\boxed{\phi(x) \equiv \frac{1}{\sqrt{2}}\left[v + \varphi_1(x) + i\,\varphi_2(x)\right]}$$

we expect this field to be a goldstone boson

We apply our strategy and look at the Lagrangian written for the shifted fields

$$\begin{aligned} D_\mu\phi(x) &= (\partial_\mu + igA_\mu)\frac{1}{\sqrt{2}}(v + \varphi_1 + i\varphi_2) = \\ &= \frac{1}{\sqrt{2}}\left[\partial_\mu\varphi_1 + i(\partial_\mu\varphi_2) + igvA_\mu + igA_\mu\varphi_1 - gA_\mu\varphi_2\right] = \\ &= \frac{1}{\sqrt{2}}\left[\partial_\mu\varphi_1 - gA_\mu\varphi_2 + i(\partial_\mu\varphi_2 + gvA_\mu + gA_\mu\varphi_1)\right] \end{aligned}$$

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \frac{1}{2}\left[(\partial_\mu\varphi_1 - gA_\mu\varphi_2)^2 + (gvA_\mu + gA_\mu\varphi_1 + \partial_\mu\varphi_2)^2\right] =$$

we separate terms that depend on v from that does not depend on v

$$= \frac{1}{2}\left[(\partial_\mu\varphi_1 - gA_\mu\varphi_2)^2 + (gA_\mu\varphi_1 + \partial_\mu\varphi_2)^2\right] + \underbrace{\frac{1}{2}g^2v^2A_\mu A^\mu}_{\substack{\text{Mass term} \\ \text{(after symm. breaking)}}} + \underbrace{gvA_\mu(gA^\mu\varphi_1 + \partial^\mu\varphi_2)}_{\substack{\text{Kinetic mixing} \\ gvA_\mu(\partial^\mu\varphi_2)}}$$

- The term $\frac{1}{2}g^2v^2A_\mu A^\mu$ is surprising ; together with the photon part it seems to describe a proca field : a massive spin-1. To support this interpretation we look at the quadratic part of the lagrangian to identify the mass spectrum . However we see that we have a "strange" object . The term $vgA_\mu(\partial^\mu\varphi_2)$ has an obscure interpretation . It is quadratic in the fields but it mixes A_μ and φ_2. It is called a Kinetic mixing.

At this point, we remember that the physics of Goldstone fields has a more transparent interpretation if we consider the exponential field variables . So we write:

$$\boxed{\phi(x) = \frac{1}{\sqrt{2}}\left[v + h(x)\right]e^{i\frac{\pi(x)}{v}}}$$

Furthermore, we remind that the theory is Gauge invariant . This means that if we consider the fields :

$$\begin{cases} \phi'(x) = e^{-i\alpha(x)}\phi(x) = e^{-i\alpha(x)}\frac{1}{\sqrt{2}}(v + h(x))\, e^{i\frac{\pi(x)}{v}} \\ A'_\mu = A_\mu(x) + \frac{1}{g}(\partial_\mu\alpha) \end{cases}$$

the theory is invariant. Since this is true for every $\alpha(x)$ we choose a special Gauge t. with:

$$\boxed{\alpha(x) = \frac{\pi(x)}{v}} \quad \underline{\text{Unitary Gauge}}$$

Therefore we have:

$$\begin{cases} \phi_u(x) = e^{-\frac{i\pi(x)}{v}}\cdot\frac{1}{\sqrt{2}}(v + h(x))\, e^{\frac{i\pi(x)}{v}} = \frac{1}{\sqrt{2}}(v + h(x)) \\ A^\mu_u(x) = A^\mu(x) + \frac{1}{gv}(\partial^\mu\pi) \end{cases}$$

(Instead of the symbol " ' " we use u to indicate the specific choice of $\alpha(x)$ that corresponds to the unitary Gauge).

To write the lagrangian in terms of ϕ_u and A^μ_u we see that:

$$\begin{cases} \phi(x) = \phi_u(x)\cdot e^{\frac{i\pi(x)}{v}} \\ D^\mu\phi = e^{\frac{i\pi(x)}{v}}(\partial^\mu\phi_u + igA^\mu_u\phi_u) \end{cases}$$

Therefore :

$$\boxed{\mathcal{L} = (\partial_\mu\phi_u + igA_{\mu_u}\phi_u)^\dagger(\partial^\mu + igA^\mu_u\phi_u) + \mu\,\phi_u^\dagger\phi_u + \lambda(\phi_u^\dagger\phi_u)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

Remembering that $\phi_u = \frac{1}{\sqrt{2}}(v + h(x))$ I could write explicitely the Kinetic and potential term

☐ <u>POTENTIAL TERM</u> :

$$\boxed{V(h) = \underbrace{\frac{1}{2}(2\mu^2)h}_{\text{massive real scalar field with } m_h^2 = 2\mu^2 = 2\lambda v^2} + \overbrace{\lambda v h^3}^{\text{cubic int.}} + \overbrace{\frac{\lambda}{4}h^4}^{\text{quartic int.}}} \longrightarrow \underline{\pi \text{ cancels out!}}$$

☐ <u>KINETIC TERM</u> :

$$\frac{1}{2}\left[\partial_\mu(h+v) + igA_\mu(h+v)\right]^\dagger\left[\partial^\mu(h+v) + igA_\mu(h+v)\right] =$$

$$= \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}g^2A_\mu A^\mu(h+v)^2 =$$

$$= \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \underbrace{\frac{1}{2}g^2v^2A_\mu A^\mu}_{\text{mass term}} + \underbrace{\frac{1}{2}g^2(h^2 + 2hv)A_\mu A^\mu}_{\text{int. between massive radial modes and } A^\mu}$$

→ <u>π cancels out</u> !

All in all, we arrive at:

$$\boxed{\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}(2\mu^2)h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4 + \\ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}g^2v^2A_\mu A^\mu + \\ &+\frac{1}{2}g^2(h^2+2hv)A_\mu A^\mu\end{aligned}}$$

i) We have a massive real scalar field h with mas $\boxed{m_h^2 = 2\mu^2 = 2\lambda v^2}$, the Higgs field with a cubic and quartic self interactions.

ii) We have a massive spin-1 field with mass $\boxed{m_A^2 = g^2v^2}$

iii) There is no Goldstone boson in the theory; its degree of freedom has been eaten by the Gauge field (the Higgs) to become massive.

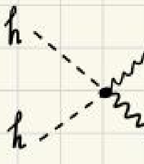
We started from a theory with 4 D.O.F. = one complex scalar field (2 D.O.F) + one massless "photon" (2 D.O.F).
We end up with a theory in the broken phase that still has 4 D.O.F. = one real scalar field (1 D.O.F) + one massive spin-1 field (3 D.O.F.).

Exercise

Write the Feynman rules describing the interaction of the Higgs field with the massive photon. We isolate

$$\begin{aligned}\mathcal{L}_{int} &= \frac{1}{2}g^2h^2A_\mu A^\mu + \frac{1}{2}g^2 2hvA_\mu A^\mu = \\ &= \frac{1}{2}g^2 g_{\mu\nu}h^2A^\mu A^\nu + g^2 v g_{\mu\nu} h A^\mu A^\nu\end{aligned}$$

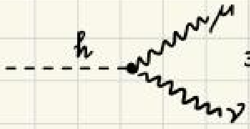
$$\mathcal{L}_{int}^{(1)} = \frac{1}{2}g^2h^2A_\mu A_\nu g^{\mu\nu} \longrightarrow$$



$$\equiv i\frac{1}{\cancel{2}}g^2g_{\mu\nu}\overset{2}{\cancel{4}} = 2ig^2g_{\mu\nu}$$

It does not vanish if v=0 indeed this is also present in scalar QED

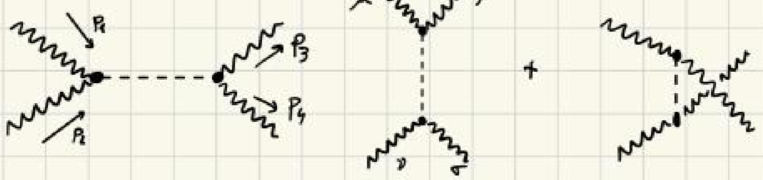
$$\mathcal{L}_{int}^{(2)} = \frac{1}{\cancel{2}}g^2\cancel{2}hv\,A_\mu A_\nu g^{\mu\nu} \longrightarrow$$



$$\equiv 2ig^2 v\, g^{\mu\nu}$$

This is peculiar of sp. symm. break. (since ∝ v)

Compute now the scattering amplitude of the scattering of 4 longitudinally polarized "photon" fields.



$$\begin{aligned}i\mathcal{M}_\mu = \left[\varepsilon_L^\mu(P_1)\varepsilon_L^\nu(P_2)\varepsilon_L^{\rho *}(P_3)\varepsilon_L^{\sigma *}(P_4)\right] &= (2igv)^2\left[\frac{g_{\mu\nu}g_{\rho\sigma}\,i}{s-m_h^2} + \frac{ig^{\mu\rho}g^{\nu\sigma}}{t-m_h^2} + \frac{ig^{\mu\sigma}g^{\nu\rho}}{u-m_h^2}\right] = \\ &= \varepsilon_L^\mu(P_1)\varepsilon_L^\nu(P_2)\varepsilon_L^{\rho *}(P_3)\varepsilon_L^{\sigma *}(P_4)\,4v^2g^4(-i)\left(\frac{g_{\mu\nu}g_{\rho\sigma}}{s-m_h^2} + \frac{g_{\mu\rho}g_{\nu\sigma}}{t-m_h^2} + \frac{g_{\mu\rho}g_{\nu\rho}}{u-m_h^2}\right)\end{aligned}$$

Let's check that in the high energy limit $i\mathcal{M}$ does not grow with energy.
In the ultra relativistic limit $\varepsilon_L^\mu(p) = \frac{p^\mu}{m_A}$

$$\longrightarrow \quad i\mathcal{M} \overset{U.R.}{\approx} \frac{4v^2 g^4(-i)}{m_A^4}\left[\frac{(p_1\cdot p_2)(p_3\cdot p_4)}{s-m_h^2} + \frac{(p_1\cdot p_3)(p_2\cdot p_4)}{t-m_h^2} + \frac{(p_1\cdot p_4)(p_2\cdot p_3)}{u-m_h^2}\right]$$

Remembering that:

$$\boxed{\begin{aligned} s &= 2m_A^2 + 2p_1\cdot p_2 = 2m_A^2 + 2p_3\cdot p_4 \\ t &= 2m_A^2 - 2p_1\cdot p_3 = 2m_A^2 - 2p_2\cdot p_4 \\ u &= 2m_A^2 - 2p_1\cdot p_4 = 2m_A^2 - 2p_2\cdot p_3 \end{aligned}} \quad \begin{aligned} &\longrightarrow \quad 2p_1\cdot p_2 = 2p_3\cdot p_4 = s - 2m_A^2 \approx s \\ &\longrightarrow \quad -2p_1\cdot p_3 = -2p_2\cdot p_4 = t - 2m_A^2 \approx t \\ &\qquad\;\; -2p_2\cdot p_3 = -2p_1\cdot p_4 = u - 2m_A^2 \approx u \end{aligned}$$

$$\begin{aligned} \longrightarrow \quad i\mathcal{M} &= \frac{-ig^4v^2}{m_A^4}\left[\frac{(s-\cancel{2m_A^2})^2}{(s-m_h^2)} + \frac{(t-\cancel{m_A^2 2})^2}{t-m_h} + \frac{(u-\cancel{m_A^2 2})^2}{u-m_h^2}\right] = \\ &= \frac{(-ig^2)\cancel{m_A^2}}{m_A^{\cancel{4}2}}\left(\frac{s^2}{s-m_h^2} + \frac{t^2}{t-m_h^2} + \frac{u^2}{u-m_h^2}\right) = \leftarrow m_A^2 = g^2v^2 \\ &= -\frac{i}{v^2}\left(\frac{s^2}{s-m_h^2} + \frac{t^2}{t-m_h^2} + \frac{u^2}{u-m_h^2}\right) \longrightarrow 0 \end{aligned}$$

if $s,t,u \gg m_h^2$, so it does not grow with energy.

<u>Comment</u>: Consider the amplitude for the "$\pi\pi$ scattering" we computed before

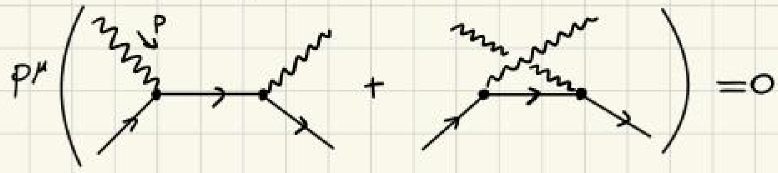
$$i\mathcal{M}_{\pi\pi\to\pi\pi} = \frac{(-i)}{v^2}\left(\frac{s^2}{s-m_h^2} + \frac{t^2}{t-m_h^2} + \frac{u^2}{u-m_h^2}\right)$$

it matches perfectly with the previous result. Thi result perfectly agrees with the interpretation that the Goldstone boson has been eaten by the massless vector field. In the high energy limit it becomes its longitudinal polarization.

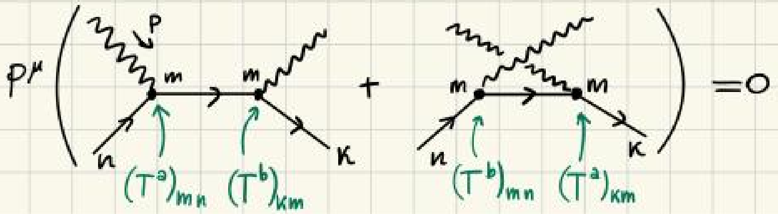
NON ABELIAN GAUGE THEORY CASE

INTRODUCTION

Consider the Compton scattering in QED. The amplitude verifies the *Ward identity*



We now try to generalize to a theory in which we have *multiple photons* and *multiple fermions*. We try to write



The amplitude has the structure (schematically):

$$iM \sim (T^b)_{km}(T^a)_{mn} - (T^a)_{km}(T^b)_{mn} = (T^bT^a)_{kn} - (T^aT^b)_{kn}$$

The new structure that we put spoils the validity of the Ward identity. Unless we have a third diagram of the kind:

In such case:

$$iM \sim (T^bT^a)_{kn} - (T^aT^b)_{kn} + if_{abc}(T^c)_{kn} \qquad \text{where } [T^a,T^b]_{kn} = if_{abc}(T^c)_{kn}$$

If the underlying structure is non-abelian, we may hope to rescue the cancellation

YANG-MILLS GAUGE THEORY

With this motivation in mind, we try to construct a non-abelian generalization of *electromagnetism*. Consider a theory with a global invariance under the transformation of a compact Lie Group G

$$\boxed{\phi^a(x) \longrightarrow \phi'^a(x) = \phi^a(x) - i\alpha_A (T^A)^a{}_b\, \phi^b(x)} \qquad (\textit{Infinitesimal transf.})$$

T^A are the generators with non-abelian Lie brackets

$$\boxed{[T^A, T^B] = if_{ABC}T^C} \qquad (f_{ABC} \text{ completely antisymmetric if the group is compact})$$

The group is compact and all *finite dimensional* representations are equivalent to unitary representations:

$$\longrightarrow (T^A)^\dagger = T^A$$

Since for a compact connected Lie group the exponential map is surjective, we have in the *finite form*:

$$\boxed{\phi(x) \longrightarrow \phi'(x) = \exp(-i\alpha_A T^A)\,\phi(x) \equiv \mathcal{U}(\vec{\alpha})\,\phi(x)}$$

The Lagrangian describing these fields will have the form:

$$\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \quad \text{and} \quad \delta\mathcal{L} = 0$$

We use electrodynamics as a model. We try to promote the global symmetry to be a local invariance of the Lagrangian.

$$\longrightarrow \quad \boxed{\phi(x) \longrightarrow \phi'(x) = \exp\left[-i\alpha_A(x)\,T^A\right]\phi(x) \equiv \mathcal{U}(x)\,\phi(x)}$$

The Lagrangian as it is is obviously not invariant under local Gauge transformations and the reason is the derivative terms.

Example

Consider a set of "N" Dirac fermions that we collect into a N-dimensional vector:

$$\Psi(x) = \begin{pmatrix} \psi^1(x) \\ \vdots \\ \psi^N(x) \end{pmatrix}$$

that transforms according to the fundamental representation of SU(N):

$$\boxed{\Psi(x) \longrightarrow \exp\left(-i \sum_{A=1}^{N-1} \alpha_A T^A\right)\Psi(x)}$$

The Lagrangian:

$$\boxed{\mathcal{L}_\Psi = \overline{\Psi}\, i\gamma^\mu \partial_\mu \Psi - m_\Psi \overline{\Psi}\,\overline{\Psi}}$$

is clearly invariant under the global symmetry infact U does not depend on x and passes through the derivative.
If we make the transformation local, this is not true anymore. In fact:

$$\overline{\Psi}\, i\gamma^\mu \partial_\mu \Psi \longrightarrow \overline{\Psi}\,\mathcal{U}^\dagger i\gamma^\mu \partial_\mu(\mathcal{U}\Psi) = \overline{\Psi}\,\mathcal{U}^\dagger i\gamma^\mu\left[(\partial_\mu \mathcal{U})\Psi + \mathcal{U}\partial_\mu\Psi\right] = \overline{\Psi}\, i\gamma^\mu(\partial_\mu\Psi) + \underbrace{\overline{\Psi}\,\mathcal{U}^\dagger(\partial_\mu\mathcal{U})\, i\gamma^\mu\Psi}_{\text{this spoils the invariance}}$$

To solve this problem, we mimic the minimal coupling prescription of QED and we introduce a covariant derivative

$$\boxed{D_\mu\phi \equiv \partial_\mu\phi + i\,G_\mu^A T^A \phi}$$

Where we introduce # dim G "photon" fields G_μ^A ("Gauge fields"). This covariant derivative must transform according to:

$$\boxed{D_\mu\phi \longrightarrow (D_\mu\phi)' = \mathcal{U}\,(D_\mu\phi)}$$

In this way terms like our previous $\overline{\Psi}\, i\gamma^\mu(D_\mu\Psi)$ would be obviously invariant.

We derive the transformation properties of the Gauge fields G_μ^A as follows:

$$(D_\mu\phi)' = \partial_\mu\phi' + i\,G_\mu'^A T^A\phi' \overset{!}{=} \mathcal{U}\,(\partial_\mu\phi + i\,G_\mu^A T^A\phi)$$

$$\partial_\mu(\mathcal{U}\phi) + i\,G_\mu'^A T^A\,\mathcal{U}\phi \overset{!}{=} \mathcal{U}\,(\partial_\mu\phi) + i\,\mathcal{U}\,G_\mu^A T^A\phi$$

$$(\partial_\mu \mathcal{U})\Phi + \cancel{\mathcal{U}(\partial_\mu\Phi)} + iG'^A_\mu T^A\mathcal{U}\Phi = \cancel{\mathcal{U}(\partial_\mu\Phi)} + i\mathcal{U}G^A_\mu T^A\Phi$$

At the operatoria level we have:

$$\longrightarrow \partial_\mu\mathcal{U} + iG'^A_\mu T^A\mathcal{U} \overset{!}{=} i\mathcal{U}G^A_\mu T^A$$

$$\longrightarrow -i(\partial_\mu\mathcal{U}) + G'^A_\mu T^A\mathcal{U} = \mathcal{U}G^A_\mu T^A$$

moltiplying by U^{-1} on the right

$$\longrightarrow G'^A_\mu T^A = +i(\partial_\mu\mathcal{U})\mathcal{U}^{-1} + \mathcal{U}G^A_\mu T^A\mathcal{U}^{-1}$$

$$\longrightarrow \boxed{G_\mu \equiv G^A_\mu T^A \,, \quad G'_\mu = +i(\partial_\mu\mathcal{U})\mathcal{U}^{-1} + \mathcal{U}G_\mu\mathcal{U}^{-1}}$$

Finite transformation of Gauge fields

We workout the transformation of G^A_μ at the infinitesimal level

$$\mathcal{U}(x) \simeq \mathbb{1} - i\alpha_A(x)T^A$$

$$\longrightarrow G'^A_\mu T^A = +i\left[\partial_\mu(\mathbb{1} - i\alpha_A T^A)\right](\mathbb{1} + i\alpha_A T^A) + (\mathbb{1} - i\alpha_A T^A)G^B_\mu T^B(\mathbb{1} + i\alpha_A T^A) =$$

$$= +(\partial_\mu\alpha_A)T^A + G^A_\mu T^A - i\alpha_A T^A G^B_\mu T^B + G^B_\mu T^B i\alpha_A T^A =$$

$$= G^A_\mu T^A + (\partial_\mu\alpha_A)T^A - i\alpha_A G^B_\mu(T^A T^B - T^B T^A) =$$

$$= G^A_\mu T^A + (\partial_\mu\alpha_A)T^A - i\alpha_A G^B_\mu if_{ABC}T^C =$$

$$= G^A_\mu T^A + (\partial_\mu\alpha_A)T^A + \alpha_C G^B_\mu f_{CBA}T^A =$$

$$= \left[G^A_\mu + (\partial_\mu\alpha_A) + f_{ABC}\alpha_B G^C_\mu\right]T^A$$

1st Key equation

$$\delta G^A_\mu \equiv G'^A_\mu - G^A_\mu \quad \longrightarrow \quad \boxed{\delta G^A_\mu = +(\partial_\mu\alpha_A) + f_{ABC}\alpha_B G^C_\mu}$$

Infinitesemal transformation of Gauge fields

<u>Comment</u>: Let us analyze what this transformation is telling us:

$$G'^A_\mu = G^A_\mu + (\partial_\mu\alpha_A) - i\underbrace{(if_{BCA})}_{\equiv (t^B)_{AC}}\alpha_B G^C_\mu \quad \longrightarrow \quad \boxed{G'^A_\mu = G^A_\mu + (\partial_\mu\alpha_A) - i\alpha_B(t^B)_{AC}G^C_\mu}$$

Example

As a concrete example, consioler the Lagrangian we studied before

$$\mathcal{L} = \bar\psi i\gamma^\mu D_\mu\psi - m_\psi\bar\psi\psi = \bar\psi i\gamma^\mu(\partial_\mu\psi) + \bar\psi i\gamma^\mu(+iG^A_\mu T^A)\psi - m_\psi\bar\psi\psi$$

$$\longrightarrow \boxed{\mathcal{L} = \bar\psi i\gamma^\mu(\partial_\mu\psi) - m_\psi\bar\psi\bar\psi - \bar\psi\gamma^\mu G^A_\mu T^A\psi}$$

this lagrangian is now invariant under local transformations.

Take now the limit in which $\alpha_A(x) = \alpha_A = \text{const}$. We downgrade back the transformation to a global symmetry. The lagrangian must be invariant, since a global symmetry is a special case of local transformation with constant parameters.

Consequently, the Gauge fields must transform under the global G according to some representation of the group. If $\alpha_A = const$ $(\partial_\mu \alpha_A = 0)$ we have:

$$\boxed{G'^A_\mu(x) = G^A_\mu(x) - i\alpha_B (t^B)_{AC}\, G^C_\mu(x)}$$

→ which means that $G^A_\mu(x)$ transform under global G transformations according to its adjoint representation

If $\alpha_A = const$, the finite transformation is $G^A_\mu T^A \longrightarrow \mathcal{U} G^A_\mu T^A \mathcal{U}^{-1}$ which is the action of the adjoint representation of the group at the level of finite group elements

$$Ad : G \longrightarrow GL(g)$$
$$Ad(g)X \longmapsto gXg^{-1}$$

If we Gauge the symmetry along the "A" direction that is $\alpha_A \to \alpha_A(x)$, then the "A" Gauge field acquires the term:

$$G'^A_\mu(x) \longrightarrow \underbrace{G^A_\mu(x) + (\partial_\mu \alpha_A(x))}_{\text{Like a photon, it shifts with } \partial_\mu\alpha_A \text{ under a Gauge transf.}} - i\alpha_B (t^B)_{AC} G^C_\mu(x)$$

To complete our construction, we need to introduce a term like $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ that describes the free dynamics of the Gauge fields. Of course this generalitation must be such that:

i) "$F_{\mu\nu}$" must be a function of G^A_μ and its derivative

ii) "$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$" must be invariant under Gauge transformations

The Key observation is the following. If we find a function $F^A_{\mu\nu}(x)$ such that:

$$\boxed{D_\mu D_\nu \phi(x) - D_\nu D_\mu \phi(x) = +i\, F^A_{\mu\nu}(x)\, T^A \phi(x)}$$

then the objects $F^A_{\mu\nu}$ will have nice transformation properties under local G.

As we have discussed,

$$\begin{cases} (D_\mu \phi)' = \mathcal{U}(D_\mu \phi) \\ (D_\mu \phi)' = D'_\mu \phi' = D'_\mu \mathcal{U}\phi \end{cases} \longrightarrow \boxed{D'_\mu = \mathcal{U} D_\mu \mathcal{U}^{-1}}$$

Consequently $D_\mu D_\nu \phi \to D'_\mu D'_\nu \phi' = (\mathcal{U} D_\mu \mathcal{U}^{-1})(\mathcal{U} D_\nu \mathcal{U}^{-1})\mathcal{U}\phi = \mathcal{U}(D_\mu D_\nu \phi)$

It follows that the left side of the previous equation transforms according to:

$$(D_\mu D_\nu \phi - D_\nu D_\mu \phi) \longrightarrow \mathcal{U}(D_\mu D_\nu \phi - D_\nu D_\mu \phi)$$

and the same must be true for the right-hand side

$$+iF^A_{\mu\nu}T^A\phi \longrightarrow \underbrace{+iF'^A_{\mu\nu}T^A\phi'}_{= iF'^A_{\mu\nu}T^A\mathcal{U}\phi} \overset{!}{=} \mathcal{U}(+iF^A_{\mu\nu}T^A\phi)$$

$$\boxed{F^A_{\mu\nu}T^A \longrightarrow F'^A_{\mu\nu}T^A = \mathcal{U}\, F^A_{\mu\nu} T^A \mathcal{U}^{-1}}$$

$F^A_{\mu\nu}$ transforms according to the adjoint of G

At the infinitesimal level, we have:

$$\mathcal{U} F^A_{\mu\nu} T^A \mathcal{U}^{-1} = (\mathbb{1} - i\alpha_A T^A) F^A_{\mu\nu} T^A (\mathbb{1} + i\alpha_B T^B) =$$

$$= F^A_{\mu\nu} T^A - i\alpha_A T^A F^B_{\mu\nu} T^B + i F^B_{\mu\nu} T^B \alpha_A T^A =$$

$$= F^A_{\mu\nu} T^A - i\alpha_A F^B_{\mu\nu} [T^A, T^B] =$$

$$= F^A_{\mu\nu} T^A - i\alpha_A F^B_{\mu\nu}\, i f_{ABC} T^C =$$

$$= F^A_{\mu\nu} T^A - i\alpha_C F^B_{\mu\nu}\, i f_{CBA} T^A =$$

$$= F^A_{\mu\nu} T^A - i\alpha_B F^C_{\mu\nu}\, i f_{BCA} T^A$$

$$\longrightarrow \boxed{F^A_{\mu\nu} \longrightarrow F'^A_{\mu\nu} = F^A_{\mu\nu} - i\alpha_B (t^B)_{AC} F^C_{\mu\nu}}$$

Infinitesimal adjoint transformation

Given these transformation properties, it is simple to write down an invariant object. Consider the quantity:

$$\boxed{tr\left[F^A_{\mu\nu} T^A F^{B,\mu\nu} T^B\right]}$$

$$\underline{tr\left(F^A_{\mu\nu} T^A F^{B,\mu\nu} T^B\right)} \longrightarrow tr\left(\mathcal{U} F^A_{\mu\nu} T^A \mathcal{U}^{-1} \mathcal{U} F^{B,\mu\nu} T^B \mathcal{U}^{-1}\right) = tr\left(\mathcal{U} F^A_{\mu\nu} T^A F^{B,\mu\nu} T^B \mathcal{U}^{-1}\right) \underset{\substack{\uparrow \\ \text{cyclic} \\ \text{property}}}{=} \underline{tr\left(F^A_{\mu\nu} T^A F^{B,\mu\nu} T^B\right)}$$

$$\longrightarrow \boxed{tr\left(F^A_{\mu\nu} T^A F^{B,\mu\nu} T^B\right) \text{ is invariant under (local) } G \text{ transformations}}$$

We can also write:

$$\boxed{tr\left(F^A_{\mu\nu} T^A F^{B,\mu\nu} T^B\right) = F^A_{\mu\nu} F^{B,\mu\nu}\, tr\left(T^A T^B\right)}$$

$tr(T^A T^B)$ is an important object. If the group is compact, whatever the repr, we can always choose a basis such that:

$$\boxed{tr\left(T^A T^B\right) \propto \delta_{AB}}$$

(in general $tr(T^A_R T^B_R) = C_R \delta_{AB}$ wher C_R is the so called index of the repr.)

In the fundamental representation, we often require:

$$\boxed{tr\left(T^A T^B\right) = \frac{1}{2}\delta_{AB}}$$

Consequently, we have that:

$$tr\left(F^A_{\mu\nu} T^A F^{B,\mu\nu} T^B\right) \propto F^A_{\mu\nu} F^{B,\mu\nu} \delta_{AB} = F^A_{\mu\nu} F^{A,\mu\nu}$$

Therefore $F^A_{\mu\nu} F^{A,\mu\nu}$ is invariant under (Gauge) transformation of the group G

We only need to find $F^A_{\mu\nu}$. We use hence the definition:

$$D_\mu D_\nu \phi - D_\nu D_\mu \phi = -i F^A_{\mu\nu} T^A \phi$$

$$D_\mu D_\nu \phi - (\mu \leftrightarrow \nu) = (\partial_\mu + i G^A_\mu T^A)(\partial_\nu + i G^A_\nu T^A)\phi - (\mu \leftrightarrow \nu) =$$

$$= \partial_\mu \partial_\nu \phi + i(\partial_\mu G^A_\nu) T^A \phi + i G^A_\nu T^A (\partial_\mu \phi) + i G^A_\mu T^A (\partial_\nu \phi) - G^A_\mu T^A G^B_\nu T^B \phi - (\mu \leftrightarrow \nu) \overset{(*)}{=}$$

There are terms that cancel out in the sum when we antisymmetrize. In particular we notice that:

$$\partial_\mu \partial_\nu \phi - \partial_\nu \partial_\mu \phi = 0$$

$$\cancel{i G^A_\nu T^A (\partial_\mu \phi)} + \cancel{i G^A_\mu T^A (\partial_\nu \phi)} - \cancel{i G^A_\mu T^A (\partial_\nu \phi)} - \cancel{i G^A_\nu T^A (\partial_\mu \phi)} = 0$$

$$(*) = i(\partial_\mu G^A_\nu) T^A \phi - G^A_\mu T^A G^B_\nu T^B \phi - i(\partial_\nu G^A_\mu) T^A \phi + G^A_\nu T^A G^B_\mu T^B \phi =$$

$$= +i(\partial_\mu G^A_\nu - \partial_\nu G^A_\mu) T^A \phi + G^B_\mu G^A_\nu (T^A T^B - T^B T^A)\phi =$$

$$= +i(\partial_\mu G^A_\nu - \partial_\nu G^A_\mu) T^A \phi + G^B_\mu G^A_\nu \, i f_{ABC} T^C \phi =$$

$$= +i(\partial_\mu G^A_\nu - \partial_\nu G^A_\mu) T^A \phi + i f_{BCA} G^B_\nu G^C_\mu T^A \phi$$

From the comparison we extract:

$$F^A_{\mu\nu} = (\partial_\mu G^A_\nu - \partial_\nu G^A_\mu) + f_{BCA} G^B_\nu G^C_\mu =$$

$$= (\partial_\mu G^A_\nu - \partial_\nu G^A_\mu) - f_{ABC} G^B_\mu G^C_\nu$$

$$\longrightarrow \boxed{F^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - f_{ABC} G^B_\mu G^C_\nu}$$ Field Strenght tensor

We arrive at the Lagrangian:

$$\boxed{\mathcal{L} = -\frac{1}{4g^2} F^A_{\mu\nu} F^{A,\mu\nu} + \mathcal{L}(\phi, D_\mu \phi)}$$

g at this stage is a free constant since:

$$F^A_{\mu\nu} F^{A,\mu\nu} = (\partial_\mu G^A_\nu - \partial_\nu G^A_\mu - f_{ABC} G^B_\mu G^C_\nu)(\partial^\mu G^{A,\nu} - \partial^\nu G^{A,\mu} - f_{ABC} G^{B,\mu} G^{C,\nu}) =$$

$$= \underbrace{(\partial_\mu G^A_\nu - \partial_\nu G^A_\mu)(\partial^\mu G^{A,\nu} - \partial^\nu G^{A,\mu})}_{\text{Kinetic terms for each field } G^A_\mu \text{ individually}} + \text{interactions}$$

if we rescale $\boxed{G^A_\mu \to g G^A_\mu}$ we make the Kinetic term for each field G^A_μ canonical (like a photon)

Consequently we arrive at the Yang-Mills Lagrangian

$$\boxed{\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{A,\mu\nu} + \mathcal{L}(\phi, D_\mu \phi)}$$

where : $$\boxed{\begin{aligned} F^A_{\mu\nu} &\equiv \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g f_{ABC} G^B_\mu G^C_\nu \\ D_\mu \phi &\equiv (\partial_\mu + i g G^A_\mu T^A)\phi \end{aligned}}$$

g has the meaning of a Gauge coupling!

"SIMPLE" GAUGE - THEORIES

The above discussion is valid, strictly speaking in the case of the so called "Simple Lie Group" (Compact simple Lie Group, therefore, if we add the additional requirement of compactness). There are different equivalent definitions of simple Lie Group.

A simple Lie Group is a connected Lie grap wose algebra is simple. A Lie algebra is simple if

i) it is non-abelian
ii) it has no non-trivial ideals (an ideal h of a Lie algebra g is a subalgebra such that $[X, H] \in h \quad \forall X \in g, H \in h$.

Equivalently, a Lie algebra is simple if and only if its adjoint representation is irreducible:

This last definition is more suitable for our purposes. The reason is that both $F^A_{\mu\nu}$ and G^A_μ transform according to the adjoint representation. Conceptually, we have the following distinction.

i) If the group is simple, the adjoint repr. is irreducible and all G^A_μ talk to each other. There is only one invariant that can be formed : $F^A_{\mu\nu} F^{A,\mu\nu}$.

All simple compact Lie algebra can be classified as:

$$\begin{cases} SU(N)\ ,\ N \geqslant 2 & \rightarrow \text{S.M.} \\ SO(N)\ ,\ N \geqslant 5 & \rightarrow \text{G.U.T.} \\ Sp(N)\ ,\ N \geqslant 1 & \\ G_2, E_4, E_6, E_7, E_8 & \rightarrow \text{String theory} \end{cases}$$

ii) If the group is not simple, the adjoint representation can be reduced into the direct sum of irreducible pieces. In this case not all the Gauge bosons talk to each other but we can "separate" them into subset according to the irreducible pieces of the adjoint.

The typical situation is the one in which G is the direct product of simple Lie group and/or abelian $U(1)$ factor:

$$\boxed{G = G_1 \times G_2 \times \dots}$$ (with G_i either simple Lie Group or an abelian $U(1)$)

In this case, each G_i comes with its own Gauge coupling:

$$\boxed{\mathcal{L} = \sum_i -\frac{1}{4g_i^2} F^{A_i}_{i\,\mu\nu} F_i^{A_i\mu\nu} + \mathcal{L}(\phi, D_\mu\phi)} \quad A_i = 1, \dots, \dim G_i$$

$$\boxed{D_\mu \phi = \partial_\mu \phi + i \sum_K G^{A_K}_{K,\mu\nu} T^{A_K}_K \phi}$$ ($T^{A_K}_K$ describes how ϕ transforms under each factor G)

INTERACTIONS IN A YANG-MILLS THEORY

$$\boxed{\mathcal{L} = \underbrace{-\frac{1}{4}F^A_{\mu\nu}F^{A,\mu\nu}}_{} + \underbrace{\mathcal{L}(\Phi, D_\mu\Phi)}_{}}$$

↓ take as example

$$\boxed{= -\frac{1}{4}\left(\partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g f_{ABC} G^B_\mu G^C_\nu\right)\left(\partial^\mu G^{A,\nu} - \partial^\nu G^{A,\mu} - g f_{ADE} G^{D,\mu} G^{E,\nu}\right)}$$

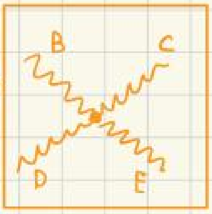
$$\begin{aligned}\mathcal{L} &= \overline{\psi} i\gamma^\mu D_\mu \psi - m_\psi \overline{\psi}\psi = \\ &= \overline{\psi} i\gamma^\mu \partial_\mu \psi - m_\psi \overline{\psi}\psi + \overline{\psi} i\gamma^\mu (igG^A_\mu T^A)\psi = \\ &= \overline{\psi} i\gamma^\mu \partial_\mu \psi - m_\psi \overline{\psi}\psi - g\overline{\psi}\gamma^\mu G^A_\mu T^A \psi = \\ &= \overline{\psi} i\gamma^\mu \partial_\mu \psi - m_\psi \overline{\psi}\psi - g\overline{\psi}_k \gamma^\mu G^A_\mu (T^A)_{km}\psi_m\end{aligned}$$

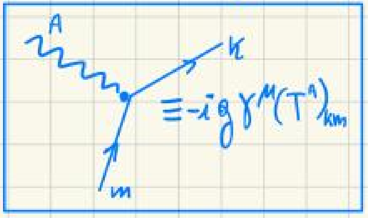
expanding we find the interaction terms:

1) $\mathcal{L}^{(1)}_{int} = \frac{1}{4}(\partial_\mu G^A_\nu - \partial_\nu G^A_\mu)\, g f_{ABC} G^{B,\mu} G^{\nu,C} \times 2$

$= +g f_{ABC} (\partial_\mu G^A_\nu) G^{B,\mu} G^{\nu,C}$ →

2) $\mathcal{L}^{(2)}_{int} = -\frac{1}{4} g^2 f_{ABC} f_{ADE} G^B_\mu G^C_\nu G^{D,\mu} G^{E,\mu}$ →





These terms describe non-linear interactions among Gauge Bosons, a peculiarity of non-abelian theories.

THE NON-ABELIAN HIGGS MECHANISM

Consider a Gauge theory based on the group SU(2). Specifically we write the Lagrangian: (N.B. $f_{ABC} = \varepsilon_{ABC}$)

$$\boxed{\mathcal{L} = (D_\mu H)^\dagger (D^\mu H) - V(H) - \frac{1}{4}F^A_{\mu\nu}F^{A,\mu\nu}}$$

where

$$\boxed{H \equiv \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \text{ is a doublet of complex scalar fields}}$$ (4 real scalar D.O.F.)

it transforms under SU(2) according to the fundamental bi-dimensional representation

$$H \longrightarrow \exp\left(-i\alpha_A \frac{\sigma^A}{2}\right) H$$

And where:

$$\boxed{D_\mu H = \partial_\mu H + igW^A_\mu \frac{\sigma^A}{2} H}$$ ($\dim G = 3 \longrightarrow 3$ Gauge fields W^A_μ)

and:

$$\boxed{F^A_{\mu\nu} = \partial_\mu W^A_\nu - \partial_\nu W^A_\mu - g\,\varepsilon_{ABC} W^B_\mu W^C_\nu}$$

The peculiarity is that the theory has the potential:

$$\boxed{V(H) \equiv -\mu^2 H^+ H - \lambda (H^+ H)^2 \quad ; \mu^2 > 0, \lambda > 0}$$

<u>Gauge invariance</u> is guaranteed by the transformations:

now local!

$$\boxed{H \longrightarrow \mathcal{U}(x) H \quad , \quad \mathcal{U}(x) = \exp\left(-i\alpha_A(x) \frac{\sigma^A}{2}\right) H}$$

with:

$$\boxed{W_\mu \longrightarrow W_\mu' = \mathcal{U}(x) W_\mu \mathcal{U}(x)^{-1} + \frac{i}{g}\left[\partial_\mu \mathcal{U}(x)\right]\mathcal{U}(x)^{-1}}$$

$\downarrow$ with $W_\mu \equiv W_\mu^A T^A = W_\mu^A \frac{\sigma^A}{2}$

The theory features spontaneous symmetry breaking. If we set $X \equiv \sqrt{H^+H}$ the vacuum manifold is described by the condition:

$$\boxed{H^+H = \frac{\mu^2}{2\lambda} = \frac{v^2}{2} \;;\quad v^2 \equiv \frac{\mu^2}{\lambda}}$$

All these vacua are connected by $SU(2)$ transformations. We make a choice and define:

$$\boxed{\langle H \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}}$$

This VEV breaks spontaneously the $SU(2)$ symmetry down to nothing since there is no subgroup of $SU(2)$ that leaves the vacuum invariant.

This spontaneous symmetry breaking should deliver three Goldston bosons, one for each broken generators. In addition, we expect one massive real scalar field.

We introduce the Goldstone matrix:

$$\boxed{H(x) = \exp\left(\frac{i\pi^a(x)\cdot\sigma^a}{2v}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h(x) \end{pmatrix}}$$

Therefore the Goldstone fields disappear from the potential since $H^+H = \frac{1}{2}(v+h)^2$.
We now introduce the unitary Gauge. $\alpha_A(x) = \frac{\pi^A(x)}{v}$. Therefore:

$$H_u(x) = \exp\left(-i\alpha_A(x)\frac{\sigma^A}{2}\right) H(x) = \exp\left(-i\alpha_A(x)\frac{\sigma^A}{2}\right)\exp\left(i\frac{\pi^a(x)\cdot\sigma^a}{2v}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h(x) \end{pmatrix}$$

$$\longrightarrow \boxed{H_u(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h(x) \end{pmatrix}}$$

Accordingly we transform the Gauge fields to:

$$\boxed{W_U^\mu = \mathcal{U} W^\mu \mathcal{U}^{-1} + \frac{i}{g}(\partial^\mu \mathcal{U})\mathcal{U}^{-1} \quad ;\; \mathcal{U} = \exp\left(\frac{i\pi^A\cdot\sigma^A}{2v}\right)}$$

By construction, the Lagrangian written in terms of $H_u(x)$ and $W_u^\mu(x)$ has the same form of the original Lagrangian. We thus drop the subscript "u" and write:

$$\boxed{\begin{gathered}\mathcal{L} = (D_\mu H)^+(D^\mu H) - V(H) - \frac{1}{4}F_{\mu\nu}^A F^{A,\mu\nu} \\ \text{with} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}\end{gathered}}$$

Theory in the broken phase

☐ SCALAR POTENTIAL:

$$V(h) = \underset{\substack{\downarrow \\ \text{mass term} \\ m_h = 2\mu^2 = 2\lambda v^2}}{\frac{1}{2}(2\mu^2)h^2} + \underbrace{\lambda v h^3 + \frac{\lambda}{4}h^4}_{\substack{\text{scalar self interactions} \\ \text{cubic and quartic}}} - \frac{\mu^4}{4\lambda}$$

☐ **KINETIC TERM** : we elaborate it as follows:

$$\begin{cases} D_\mu H = \left(\partial_\mu + i g W_\mu^A \frac{\sigma^A}{2}\right) H = \partial_\mu H + i g W_\mu^A \frac{\sigma^A}{2} H \\ (D_\mu H)^\dagger = \partial_\mu H^\dagger - i g H^\dagger W_\mu^A \frac{\sigma^A}{2} \end{cases}$$

we compute:

$$(D_\mu H)^\dagger (D^\mu H) = \left(\partial_\mu H^\dagger - i g H^\dagger \frac{W_\mu^A \sigma^A}{2}\right)\left(\partial^\mu H + i g W^{A,\mu} \frac{\sigma^A}{2} H\right) =$$

$$= (\partial_\mu H^\dagger)(\partial^\mu H) - \frac{ig}{2} W_\mu^A H^\dagger \sigma^A (\partial^\mu H) + \frac{ig}{2} W_\mu^A (\partial^\mu H^\dagger) \sigma^A H + \frac{g^2}{4} W_\mu^A W^{B,\mu} H^\dagger \sigma^A \sigma^B H$$

Consider the terms one by one:

- $(\partial_\mu H^\dagger)(\partial^\mu H) = \frac{1}{2}(\partial_\mu h)(\partial^\mu h)$ Kinetic term for $h(x)$
- $i \frac{g}{2} W_\mu^A \left[(\partial^\mu H^\dagger)\sigma^A H - H^\dagger \sigma^A (\partial^\mu H)\right] = \frac{i g W_\mu^A}{2} \frac{1}{2}\left[(0, \partial^\mu h)\sigma^A \begin{pmatrix} 0 \\ v+h \end{pmatrix} - (0, v+h)\sigma^A \begin{pmatrix} 0 \\ \partial^\mu h \end{pmatrix}\right] = 0$

  (for $A=1,2$ each term vanishes ; if $A=3$ $\frac{1}{2}\left[(0,\partial^\mu h)\begin{pmatrix} 0 \\ -(v+h) \end{pmatrix} - (0, v+h)\begin{pmatrix} 0 \\ -\partial^\mu h \end{pmatrix}\right] = 0$)

- $\frac{g^2}{4} W_\mu^A W^{B,\mu} H^\dagger \sigma^A \sigma^B H = \frac{g^2}{4} W_\mu^A W^{B,\mu} H^\dagger (i \varepsilon_{ABC} \sigma^C + \delta_{AB}) H = \frac{g^2}{4} W_\mu^A W^{B,\mu} \delta_{AB} \frac{1}{2}(h+v)^2 =$

  $$= \frac{g^2}{8} W_\mu^A W^{B,\mu} \delta_{AB} (v^2 + h^2 + 2hv) =$$

  $$= \frac{g^2}{8} W_\mu^A W^{A,\mu} v^2 + \frac{g^2}{8}(h^2 + 2hv) W_\mu^A W^{A,\mu}$$

So we arrive at the Lagrangian:

$$\boxed{\begin{aligned} \mathcal{L} = {} & \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}(2\mu^2) h^2 - v\lambda h^3 - \frac{\lambda}{4} h^4 + \\ & - \frac{1}{4} F_{\mu\nu}^A F^{A,\mu\nu} + \frac{1}{2}\left(\frac{g^2 v^2}{4}\right) \sum_{A=1}^{3} W_\mu^A W^{A,\mu} + \\ & + \frac{1}{2}\left(\frac{g^2}{4}\right)(h^2 + 2hv) \sum_{A=1}^{3} W_\mu^A W^{A,\mu} \end{aligned}}$$

i) The Higgs boson is a massive scalar field with mass $\boxed{m_h^2 = 2\mu^2 = 2\lambda v^2}$

ii) All SU(2) Gauge bosons become massive. Their mass is $\boxed{m_W^2 = \frac{1}{4} g^2 v^2}$

iii) There are no Goldstone bosons in the spectrum: the 3 Goldstone bosons at the start are "eaten" by the Gauge fields that are now massive.

Standard Model :

$$SU(3)_{\underset{\underset{\text{"Colour"}}{\uparrow}}{c}} \times \overbrace{SU(2)_L \times U(1)_{\underset{\underset{\text{Hyper charge}}{\downarrow}}{Y}}}^{U(1)_{EH}}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

8 3+ 1 bosons

$$\left\{\begin{array}{l}\text{3 Massive Gauge} \\ \text{1 Gauge massless}\end{array}\right.$$

electro weak unification